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MULTI-SOLITONS AND RELATED SOLUTIONS FOR THE WATER-WAVES SYSTEM

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ABSTRACT. The main result of this work is the construction of multi-solitons solutions that is to say solutions that are time asymptotics to a sum of decoupling solitary waves for the full water waves system with surface tension.

1. Introduction

We consider the motion of an irrotational, incompressible fluid with constant density. We consider the situation where the fluid domain is a strip with a rigid bottom and a free surface:

$$\Omega_t = \{Y = (X, z) \in \mathbb{R}^{d+1} \, : \, -H < z < \eta(t, X)\},$$

where t is the time, $d = 1, 2$ is the horizontal dimension, H is a parameter defining the fixed bottom $z = -H$ and $z = \eta(t, X)$ is the equation of the unknown free surface at time t. We shall say that we are in the one dimensional case when $X = x \in \mathbb{R}$ and in the two-dimensional case when $X = (x, y) \in \mathbb{R}^2$. A large part of the paper will be devoted to the one-dimensional situation. We denote by u the speed of the fluid, since the motion is irrotational, it is given by $u = \nabla_Y \Phi = (\nabla_X \Phi, \partial_z \Phi)$ for some scalar function Φ and hence we find that inside the fluid domain Ω_t,

$$\nabla_Y \cdot u = \Delta_Y \Phi = (\Delta_X + \partial_z^2)\Phi = 0\,. \tag{1.1}$$

On the boundaries of Ω_t, we make the usual assumption that no fluid particles cross the boundary. At the bottom of the fluid this reads

$$\partial_z \Phi(t, X, -H) = 0 \tag{1.2}$$

and on the free surface, this yields the kinematic condition

$$\partial_t \eta(t, X) + \nabla_X \Phi(t, X, \eta(t, X)) \cdot \nabla_X \eta(t, X) - \partial_z \Phi(t, X, \eta(t, X)) = 0\,. \tag{1.3}$$

On the free surface, we also need to impose the pressure, taking into account the surface tension and using the Bernouilli law to eliminate the pressure, we find that:

$$\partial_t \Phi(t, X, \eta(t, X)) + \frac{1}{2}|\nabla_Y \Phi(t, X, \eta(t, X))|^2 + g\eta(t, X) = b\nabla_X \cdot \frac{\nabla_X \eta(t, X)}{\sqrt{1 + |\nabla_X \eta(t, X)|^2}}\,. \tag{1.4}$$

The number b is the surface tension coefficient and g is the gravitational constant. The term $g\eta(t, X)$ is the trace of the gravitational force gz on the free surface.

It is classical to rewrite the system (1.1), (1.3), (1.4) as a system involving unknowns defined on the free surface only [31]. For that purpose, let us define the following Dirichlet-Neumann operator: for given $\eta(X)$ $\varphi(X)$, we define $\Phi(X, z)$ as the (well-defined) solution of the elliptic boundary value problem

$$(\Delta_X + \partial_z^2)\Phi = 0, \quad \text{in} \quad \{(X, z) \, : \, -H < z < \eta(X)\}, \quad \Phi(X, \eta(X)) = \varphi(X), \quad \partial_z \Phi(X, -H) = 0,$$

and we define the Dirichlet-Neumann operator as

$$\begin{aligned}(G[\eta]\varphi)(X) &:= (\partial_z\Phi - \nabla_X\eta\cdot\nabla_X\Phi)|_{z=\eta(X)}\\ &= \sqrt{1+|\nabla_X\eta|^2}(\nabla_{X,z}\Phi\cdot\mathbf{n})|_{z=\eta(X)},\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector on the free surface at the point $z=\eta(X)$.

This allows to rewrite the system only in terms of the unknowns

$$(\eta(t,X),\varphi(t,X)) := (\eta(t,X),\Phi(t,X,\eta(t,X)))\,.$$

In the one-dimensional case, the 1D water-wave problem can thus be written as

$$\begin{cases}\partial_t\eta = G[\eta]\varphi\\ \partial_t\varphi = -\dfrac{1}{2}|\partial_x\varphi|^2+\dfrac{1}{2}\dfrac{(G[\eta]\varphi+\partial_x\varphi\partial_x\eta)^2}{1+|\partial_x\eta|^2}-g\eta+b\partial_x\Big(\dfrac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big)\end{cases}\tag{1.5}$$

By introducing the notations $U=(\eta,\varphi)^t$ and

$$\mathcal{F}(U)=\Big(G[\eta]\varphi,\quad -\frac{1}{2}|\partial_x\varphi|^2+\frac{1}{2}\frac{(G[\eta]\varphi+\partial_x\varphi\partial_x\eta)^2}{1+|\partial_x\eta|^2}-g\eta+b\partial_x\Big(\frac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\Big)\Big)^t,$$

we shall write the water-wave system (1.5) in the abstract form

$$\partial_t U=\mathcal{F}(U).\tag{1.6}$$

We know from [3] that for suitable parameters g, b and h, there exist solitary wave solutions $Q_c(x-ct)=(\eta_c(x-ct),\varphi_c(x-ct))^t$ at speed $c\sim\sqrt{gH}$. Here is a precise statement.

Theorem 1.1 (Amick-Kirchgässner [3])**.** *Suppose that*

$$\alpha=\frac{gH}{c^2}=1+\varepsilon^2,\quad \beta=\frac{b}{Hc^2}>\frac{1}{3}\,.\tag{1.7}$$

Then there exists ε_0 such that for every $\varepsilon\in(0,\varepsilon_0)$ (which fixes the speed) there is a solution of (1.5) under the form

$$Q_c(x-ct)=\big(\eta_c(x-ct),\varphi_c(x-ct)\big)^t=\Big(H\eta_\varepsilon(H^{-1}(x-ct)),\,cH\varphi_\varepsilon(H^{-1}(x-ct))\Big)^t$$

with

$$\eta_\varepsilon(x)=\varepsilon^2\Theta_1(\varepsilon x,\varepsilon),\quad \varphi_\varepsilon(x)=\varepsilon\Theta_2(\varepsilon x,\varepsilon),$$

where Θ_1 and Θ_2 satisfy:

$$\exists\, d>0,\quad \forall\,\alpha\geq 0,\quad \exists\, C_\alpha>0,\quad \forall\,(x,\varepsilon)\in\mathbb{R}\times(0,\varepsilon_0),\quad |(\partial_x^\alpha\Theta_1)(x,\varepsilon)|\leq C_\alpha e^{-d|x|}$$

and

$$\exists\, d>0,\quad \forall\,\alpha\geq 1,\quad \exists\, C_\alpha>0,\quad \forall\,(x,\varepsilon)\in\mathbb{R}\times(0,\varepsilon_0),\quad |(\partial_x^\alpha\Theta_2)(x,\varepsilon)|\leq C_\alpha e^{-d|x|}\,.$$

Moreover Θ_1 is even and Θ_2 is odd.

The main aim of this paper is to construct multi-solitons type solutions for the water-waves system (1.5). This means that we want to construct a solution of (1.5) that tends to a sum of solitary waves with different speeds when t goes to infinity. For the sake of readability of the paper, we shall only consider the case of two solitary waves, the extension to an arbitrary numbers is straightforward (our proof will not use any particular symmetry or specificity related to the 2-solitons case). The construction of such solutions for semi-linear equations like the KdV equation or the Nonlinear-Schrödinger equation has been intensively studied recently, we refer for example to [18, 19, 16, 9, 8, 22]. We also refer to an earlier closely related work by Merle [20] which seems to initiate this line of research.

In this paper, we shall show that the construction can be also performed for quasilinear equations (or even fully nonlinear equations) by focusing on the physically interesting example of the water-waves system (1.5). The new difficulty that arises in the case of quasilinear or fully nonlinear equations like (1.5) is that the only well-posedness result which is known in the vicinity of the solitary wave is a local well-posedness result in H^s for s sufficiently large which comes from the high order energy method. Note that the global or almost global existence results like the ones of [29, 30], [11, 12] are obtained in a regime where solitary waves do not exist and that the local existence results for rough data as obtained in [2] still require a regularity much higher than the one which is controlled by the Hamiltonian.

This makes the perturbative analysis more delicate and requires new ideas with respect to the semi-linear setting.

We thus consider two solitons $Q_{c_1}(x-c_1t)$ and $Q_{c_2}(x-h-c_2t)$ of (1.5) with $c_1 < c_2$. We suppose that c_1 and c_2 satisfy (1.7) with suitable choices of the small parameters $\varepsilon_{1,2}$. We also suppose that $h > 0$ is large enough. We define

$$M(t,x) := Q_{c_1}(x-c_1t) + Q_{c_2}(x-h-c_2t) \tag{1.8}$$

as the two-soliton function. We will focus on the case where each solitary wave is stable in the following sense. Under our assumptions (1.7) on the speed c of a solitary wave, it was proven in [21] that for sufficiently small corresponding parameter ε, the solitary wave Q_c is stable in the sense that the second derivative of the Hamiltonian at the solitary wave restricted to a natural co-dimension 2 subspace is positive. We shall assume that the speeds c_1, c_2 are such that this property is verified (see Proposition 3.6). Our main result reads:

Theorem 1.2. *Let us fix $s \geq 0$. Suppose that the speeds $c_1 < c_2$ satisfy (1.7) with parameters ε_1, ε_2. Define M by (1.8). Then there exists ε^* such that for $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon^*]$ and h sufficiently large, we have that there exists a (semi) global solution $U(t) = (\eta, \varphi)^t$ to the water-wave system (1.5) satisfying*

$$U - M \in \mathcal{C}_b([0,\infty);\, H^s(\mathbb{R}) \times H^s(\mathbb{R}))$$

and

$$\lim_{t\to+\infty} \|U(t) - M(t)\|_{H^s} = 0.$$

The assumption that $\varepsilon_{1,2}$ are sufficiently small will be only used in order to ensure the existence of smooth exponentially decreasing solitary waves (given by Theorem 1.1 for example) and that Proposition 3.6 below holds true. Indeed, for the construction of multi-solitary waves that we shall perform, the main ingredients that are required are besides the existence of smooth localized solitary waves, the stability property given by Proposition 3.6 of each solitary wave and the local well-posedness in H^s for s sufficiently large of the nonlinear system with the existence of a quasilinear hyperbolic system type energy estimate. Instead of the smallness assumption on ε in Theorem 1.2, we could assume the existence of the solitary waves (solitary waves can also be obtained by variational methods for example, [6]) and that each of them satisfies the stability property of Proposition 3.6.

We have focused on water waves with surface tension, nevertheless, since the existence of solitary waves is also known (see [13] for example) and since some of them are linearly stable, [23] (note that nevertheless an estimate like the one of Proposition 3.6 does not hold in this case, the number of negative directions of L would be infinite), it could be possible to perform a related construction for water waves without surface tension.

Finally, let us point out that the assumption that h is sufficiently large is only used in order to get a solution on $[0, +\infty[$, an equivalent statement would be to take $h = 0$ and to get a multi-soliton solution on the interval $[T_0, +\infty[$ with T_0 sufficiently large.

There are two main steps in the proof of Theorem 1.2. The first step is to construct a smooth approximate solution of (1.5) that tends exponentially fast to M as t goes to infinity. This approximate solution U^a is under the form:

$$U^a = M + \sum_{l=1}^{N} V^l$$

where each V^l is smooth and verifies the crucial property that it is decaying in H^s like $e^{-l\epsilon_0(c_2-c_1)t}$ for some ϵ_0. Each V^l solves a linear problem with source term. The existence of some V^l with this decay property will be proven by using the spectral properties of the linearization of the water waves system (1.5) about each solitary wave.

Once the approximate solution is sufficiently accurate (i.e for N sufficiently large) which basically means that the remainder term in the equation has a sufficiently fast decay in time, we shall construct an exact solution as a sum of the approximate solution and remainder term that solves a nonlinear equation. The main difficulty is to prove the existence of a solution on $[0, +\infty)$ for this problem having at hand only a local Cauchy Theory in H^s for s large.

Note that this kind of iterated constructions is related to Grenier's argument [14] in order to prove that linear instability implies nonlinear instability for quasilinear equations that has been used in [24].

The main arguments that we shall use to prove Theorem 1.2 can also be used in order to sharpen the transverse instability result proven in [24] and construct for the two-dimensional water-waves system that is to say when the fluid domain is

$$\Omega_t = \{(X, z) \in \mathbb{R}^3,\ -H < z < \eta(t, X)\},$$

a solution on $[0, +\infty)$ of the system which is different from the solitary wave (and all its translates) and converge to the solitary wave as time goes to infinity. The result that we shall prove is the following.

Theorem 1.3. *Let us fix $s \geq 0$. Suppose that c satisfies (1.7). For ε sufficiently small there exists a global solution U of the 2-D water waves system with initial data U_0 satisfying $U - Q_c \in \mathcal{C}_b([0, \infty); H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2))$. Moreover, one has*

$$\partial_y U_0 \neq 0 \tag{1.9}$$

and

$$\lim_{t \to +\infty} \|U(t, x, y) - Q_c(x - ct)\|_{H^s} = 0\,.$$

We shall recall the formulation of the 2D water-waves system in Section 2.

Remark 1.4. *By the remark after [25, Theorem 1.5] we know that this theorem implies the transverse instability of the solitary wave.*

This result can be compared to classical results about the existence of strongly stable manifolds for ordinary differential equations or semilinear partial differential equations. Results as in Theorem 1.3 were in particular, obtained for semilinear partial differential equations in [10, 7] for example and also in [25] for the KP-I equation. As previously, the main difficulty for the proof of this result comes from the fact that the water-waves system is not semilinear. The proof of Theorem 1.3 also relies on the construction of a well-chosen approximate solution and of a remainder that solves a nonlinear system. The construction of the approximate solution relies on spectral information and semi-group estimates that are contained in [24]. Consequently, we shall first present the proof of Theorem 1.3 in Section 2, this allows us to essentially focus on the construction of a remainder that is defined on the whole time interval $[0, +\infty[$. These arguments will be also useful for the proof of Theorem 1.2.

The paper is organized as follows. The next section will be devoted to the various steps of the proof of Theorem 1.3. In Section 3, we collect some useful bounds for the Dirichlet-Neumann map and we prove the key coercivity property of Proposition 3.6. We shall study in Section 4 the error that produces in the equation the sum M of two solitary waves. Then in Section 5, we shall construct a suitable approximate solution. This will be the most difficult part in the proof, the crucial step will be to prove that the fundamental solution of the linearized equation about M has a sufficiently small exponential growth. The final step of the proof of Theorem 1.2 is given in Section 6.

2. Proof of Theorem 1.3

In this section we consider the two-dimensional water-waves system. The fluid domain at time t is defined by

$$\Omega_t = \{(X, z) \in \mathbb{R}^3 \mid -H < z < \eta(t, X)\}$$

where $X = (x, y)$, $H > 0$ is a constant and $\eta(t, X)$ is the free surface at time t. We use the Zakharov formulation recalled in the introduction [31, 17] to write the system for the unknowns $\eta(t, X)$ which is the free-surface and $\varphi(t, X)$ which is the trace on the free surface of the velocity potential as:

$$\begin{cases} \partial_t \eta = G[\eta]\varphi, \\ \partial_t \varphi = -\dfrac{1}{2}|\nabla_X \varphi|^2 + \dfrac{1}{2}\dfrac{(G[\eta]\varphi + \nabla_X \varphi \cdot \nabla_X \eta)^2}{1 + |\nabla_X \eta|^2} - g\eta + b\nabla_X \cdot \dfrac{\nabla_X \eta}{\sqrt{1 + |\nabla_X \eta|^2}} \end{cases} \tag{2.1}$$

where again b is the coefficient of surface tension, g is the gravity coefficient and G is the Dirichlet-Neumann operator.

Since we study here a single solitary wave with speed c, we change frame (x, y, z) to $(x - ct, y, z)$. This leads to a new system

$$\begin{cases} \partial_t \eta = c\partial_x \eta + G[\eta]\varphi, \\ \partial_t \varphi = c\partial_x \varphi - \dfrac{1}{2}|\nabla_X \varphi|^2 + \dfrac{1}{2}\dfrac{(G[\eta]\varphi + \nabla_X \varphi \cdot \nabla_X \eta)^2}{1 + |\nabla_X \eta|^2} - g\eta + b\nabla_X \cdot \dfrac{\nabla_X \eta}{\sqrt{1 + |\nabla_X \eta|^2}}. \end{cases} \tag{2.2}$$

We also perform the following change of variables

$$\eta(t, X) = H\tilde{\eta}(\frac{c}{H}t, \frac{1}{H}X), \qquad \varphi(t, X, z) = cH\tilde{\varphi}(\frac{c}{H}t, \frac{1}{H}X) \tag{2.3}$$

and simply note $(\tilde{\eta}, \tilde{\varphi})$ again as (η, φ) to have the dimensionless water-waves system

$$\begin{cases} \partial_t \eta = \partial_x \eta + G[\eta]\varphi, \\ \partial_t \varphi = \partial_x \varphi - \dfrac{1}{2}|\nabla_X \varphi|^2 + \dfrac{1}{2}\dfrac{(G[\eta]\varphi + \nabla_X \varphi \cdot \nabla_X \eta)^2}{1 + |\nabla_X \eta|^2} - \alpha\eta + \beta\nabla_X \cdot \dfrac{\nabla_X \eta}{\sqrt{1 + |\nabla_X \eta|^2}} \end{cases} \tag{2.4}$$

with

$$\alpha = \frac{gH}{c^2}, \qquad \beta = \frac{b}{Hc^2}.$$

Observe that there is a slight abuse of notation, since in (2.4) the map $G[\eta]$ is defined as above but with $H = 1$. If we note $U = (\eta, \varphi)^t$, the system (2.4) can be rewritten as

$$\partial_t U = \mathcal{F}(U) \tag{2.5}$$

with

$$\mathcal{F}(U) = \begin{pmatrix} \partial_x \eta + G[\eta]\varphi \\ \partial_x \varphi - \dfrac{1}{2}|\nabla_X \varphi|^2 + \dfrac{1}{2}\dfrac{(G[\eta]\varphi + \nabla_X \varphi \cdot \nabla_X \eta)^2}{1 + |\nabla_X \eta|^2} - \alpha\eta + \beta\nabla_X \cdot \dfrac{\nabla_X \eta}{\sqrt{1 + |\nabla_X \eta|^2}} \end{pmatrix}.$$

The solitary wave Q_c of the original system (2.1) becomes a stationary wave solution for the water-waves system (2.4) that we shall denote in this section by $Q_\varepsilon(x) = (\eta_\varepsilon(x), \varphi_\varepsilon(x))^t$ or simply as $Q(x)$. We want to show that there exists a global solution of system (2.4) near the solitary wave.

2.1. **The linearized operator.** As in [24], we linearize the water-waves system (2.4) around the solitary wave $Q_\varepsilon = (\eta_\varepsilon, \varphi_\varepsilon)^t$. Let us set

$$Z_\varepsilon := Z[Q_\varepsilon] = \frac{G[\eta_\varepsilon]\varphi_\varepsilon + \partial_x\eta_\varepsilon\partial_x\varphi_\varepsilon}{1+|\partial_x\eta_\varepsilon|^2}, \quad v_\varepsilon := \partial_x\varphi_\varepsilon - Z_\varepsilon\partial_x\eta_\varepsilon,$$

and

$$P_\varepsilon\eta := \beta\nabla_X \cdot \left[\frac{\nabla_X\eta}{(1+(\partial_x\eta_\varepsilon)^2)^{\frac{1}{2}}} - \frac{(\nabla_X\eta_\varepsilon\cdot\nabla_X\eta)\nabla_X\eta_\varepsilon}{(1+(\partial_x\eta_\varepsilon)^2)^{\frac{3}{2}}}\right].$$

Note that since $G[\eta_\varepsilon]\varphi_\varepsilon = -\partial_x\eta_\varepsilon$, we obtain that Z_ε and v_ε are decaying exponentially together with all their derivatives thanks to Theorem 1.1 (φ_ε occurs with a derivative in v_ε). The linearization of (2.4) about Q_ε reads

$$\partial_t U = J\Lambda U, \tag{2.6}$$

where $U = (\eta, \varphi)^t$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a skew-symmetric matrix and

$$\Lambda = \begin{pmatrix} -P_\varepsilon + \alpha + Z_\varepsilon G[\eta_\varepsilon](Z_\varepsilon\cdot) + Z_\varepsilon\partial_x v_\varepsilon & (v_\varepsilon - 1)\partial_x - Z_\varepsilon G[\eta_\varepsilon] \\ -\partial_x((v_\varepsilon-1)\cdot) - G[\eta_\varepsilon](Z_\varepsilon\cdot) & G[\eta_\varepsilon] \end{pmatrix}$$

is a symmetric operator on $L^2 \times L^2$. The formula for the differential of the Dirichlet-Neumann operator with respect to η which allows to obtain the above expression is recalled in Proposition 3.1 (5) below. As in [17], we can get a simpler form of the linearized system by the change of unknowns

$$V_1 = \eta, \quad V_2 = \varphi - Z_\varepsilon\eta. \tag{2.7}$$

We get for $V = (V_1, V_2)^t$ the system

$$\partial_t V = JLV, \tag{2.8}$$

where L is still a symmetric defined by

$$L = \begin{pmatrix} -P_\varepsilon + \alpha + (v_\varepsilon - 1)\partial_x Z_\varepsilon & (v_\varepsilon - 1)\partial_x \\ -\partial_x((v_\varepsilon - 1)\cdot) & G[\eta_\varepsilon] \end{pmatrix}.$$

Notice that since Q_ε does not depend on y, the study of system (2.8) can be simplified by using Fourier transform in y. In fact, if for some $k \in \mathbb{R}$,

$$V(x,y) = e^{iky}W(x),$$

then

$$LV = e^{iky}L(k)W$$

with a symmetric operator $L(k)$ defined by

$$L(k) = \begin{pmatrix} -P_{\varepsilon,k} + \alpha + (v_\varepsilon - 1)\partial_x Z_\varepsilon & (v_\varepsilon - 1)\partial_x \\ -\partial_x((v_\varepsilon - 1)\cdot) & G_{\varepsilon,k} \end{pmatrix}. \tag{2.9}$$

Here

$$P_{\varepsilon,k}u = \beta\left\{\partial_x\left[(1+(\partial_x\eta_\varepsilon)^2)^{-\frac{3}{2}}\partial_x u\right] - k^2(1+(\partial_x\eta_\varepsilon)^2)^{-\frac{1}{2}}u\right\},$$

and $G_{\varepsilon,k}$ is such that

$$G[\eta_\varepsilon](f(x)e^{iky}) = e^{iky}G_{\varepsilon,k}(f(x)).$$

2.2. **The eigenvalue problem $JL(k)U = \sigma U$.** We shall need some results about the spectrum and the eigenvalues of $JL(k)$ seen as an unbounded operator on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \times H^1(\mathbb{R})$ which are essentially contained in [24].

Lemma 2.1. *We have the following spectral properties of the operators $JL(k)$.*

- *There exists ε^* such that for every $\varepsilon \in (0, \varepsilon^*]$, the solitary wave Q_ε is spectrally stable against one-dimensional perturbation: $\sigma(JL(0)) \subset i\mathbb{R}$ where $\sigma(\cdot)$ denotes the spectrum.*
- *For any $k \neq 0$, the essential spectrum of $JL(k)$ is such that $\sigma_{ess}(JL(k)) \subset i\mathbb{R}$ and $JL(k)$ has at most one simple eigenvalue of positive real part.*
- *If σ is an eigenvalue of $JL(k)$ then so is $-\sigma$.*
- *If σ is an eigenvalue of $JL(k)$, $k \neq 0$, with non-zero real part then $\sigma \in \mathbb{R}$.*

Note that the combination of above properties of the eigenvalues also yield that there is at most one eigenvalue of negative real part for $JL(k)$, $k \neq 0$.

Proof. These statements are already contained in Lemma 5.1 in [24]. Note that the first statement in the above lemma was obtained as a consequence of the work of Mielke [21]. The only additional point is to notice that thanks to the reversibility symmetry of the water waves system and the symmetry of the solitary wave one has that if $U(x) = (\eta(x), \varphi(x))$ is an eigenfunction of $JL(k)$ corresponding to an eigenvalue σ then $\tilde{U}(x) = (\eta(-x), -\varphi(-x))^t$ is an eigenfunction corresponding to the eigenvalue $-\sigma$ (this symmetry of the spectrum could also be obtained by using the Hamiltonian structure). The last point is a consequence of the symmetry of the spectrum and the fact that there is at most one simple eigenvalue of positive real part. $\square$

Thanks to Lemma 2.1 and in particular the symmetry of the spectrum, we have the following counterparts of [24, Proposition 5.2] and [24, Theorem 5.3] respectively.

Lemma 2.2. *Consider the eigenvalue problem*

$$JL(k)U = \sigma U, \quad U \in H^2(\mathbb{R}) \times H^1(\mathbb{R}). \tag{2.10}$$

Then :
(1) $\exists K > 0$ such that if $|k| > K$, there is no solution of (2.10) satisfying $\sigma < 0$.
(2) $\exists M > 0$ such that if $|k| \leq K$, there is no solution of (2.10) satisfying $\sigma \leq -M$.

From now on, we shall only consider solitary waves Q_ε with $\varepsilon \in (0, \varepsilon^*]$ as stated in Theorem 1.3 so that the above spectral properties are matched.

Proposition 2.3. *For $\varepsilon \in (0, \varepsilon^*]$, there exists $\sigma > 0$, $k \neq 0$ and a non-trivial $U \in H^2 \times H^1$ such that*

$$JL(k)U = -\sigma U.$$

Finally, we also have as a consequence of the above results:

Lemma 2.4. *For every (σ_0, k_0) satisfying $k_0 \neq 0$, Re $\sigma_0 < 0$ and $\sigma_0 \in \sigma(JL(k_0))$, the set $\{(\sigma, k) \,|\, \sigma \in \sigma(JL(k))\}$ near (σ_0, k_0) is the graph of an analytic curve $k \mapsto \sigma(k)$ with $\sigma(k)$ a real simple eigenvalue of $JL(k)$. Moreover, we can select an eigenvector $V(k)$ of $JL(k)$ depending analytically on k.*

Proof. Suppose that $\sigma_0 \in \sigma(JL(k_0))$ with $k_0 \neq 0$, Re $\sigma_0 < 0$. From Lemma 2.1, σ_0 is necessarily a simple real eigenvalue. Let $V_0 \in H^2 \times H^1$ an eigenvector, $JL(k_0)V_0 = \sigma_0 V_0$. Consider the map $F(V, k, \sigma) \equiv JL(k)V - \sigma V$. Then $D_{V,\sigma}F(V_0, k_0, \sigma_0)(U, \mu) = JL(k_0)U - \sigma_0 U - \mu V_0$. Thanks to Lemma 2.1, $JL(k_0) - \sigma_0$ is Fredholm with index zero and its kernel is one-dimensional. This implies that the kernel of $D_{V,\sigma}F(V_0, k_0, \sigma_0)$ is also one dimensional. Indeed, if (U, μ) is such that $D_{V,\sigma}F(U_0, k_0, \sigma_0)(U, \mu) = 0$ then $(JL(k_0) - \sigma_0)^2 U = 0$ which, thanks to Lemma 2.1, implies that

$U = \lambda V_0$, $\lambda \in \mathbb{R}$. This in turn implies that $\mu = 0$ and thus the kernel of $D_{V,\sigma}F(V_0, k_0, \sigma_0)$ is spanned by $(V_0, 0)$. Once again, thanks to Lemma 2.1, we obtain that V_0 is not in the image of $JL(k_0) - \sigma_0$ (otherwise σ_0 would not be a simple eigenvalue of $JL(k_0)$). Therefore $D_{V,\sigma}F(V_0, k_0, \sigma_0)$ is surjective. We are in position to apply the simplest form of the Lyapounov-Schmidt method. This ends the proof of Lemma 2.4. □

2.3. **Construction of the approximate solution.** Let us define for $U = (\eta, \varphi)^t$, the norms

$$\|U(t)\|_{E^s} = \sum_{0 \le \alpha+\beta+\gamma \le s} \|\partial_t^\alpha \partial_x^\beta \partial_y^\gamma U(t, \cdot)\|_{L^2(\mathbb{R}^2)} .$$

Proposition 2.5. *There exists $U^0(t, x, y) \in \cap_{s \ge 0} E^s$ satisfying*

$$\partial_t U^0 = J\Lambda U^0 \tag{2.11}$$

such that for every $s \ge 0$, every $\epsilon_0 > 0$ one has

$$\|U^0(t)\|_{E^s} \le c_{s,\epsilon_0} e^{-(\sigma_0 - \epsilon_0)t}, \qquad t \ge 0$$

where $-\sigma_0$ is the smallest possible eigenvalue of $JL(k)$ (and thus σ_0 is the largest possible eigenvalue of $JL(k)$).

Note that we use the notations ϵ_0 and ε_0 for different parameters.

Proof of Proposition 2.5. The construction is close to the one of Proposition 6.1 in [24]. The situation is even simpler here since we need less precise information on the asymptotic behavior. We already know from (2.7) that it is equivalent to solve (2.11) for U^0 and

$$\partial_t V^0 = JLV^0 \tag{2.12}$$

with

$$U^0 = PV^0, \quad P = \begin{pmatrix} 1 & 0 \\ Z_\varepsilon & 1 \end{pmatrix}.$$

for V^0.

First of all, one should locate some negative eigenvalue of $JL(k)$. We already know from Proposition 2.3 that there exists $k \ne 0$ such that $JL(k)$ has an eigenvalue $-\delta < 0$. Moreover, thanks to Lemma 2.2, we know that the negative eigenvalues $-\sigma$ of $JL(k)$ can only be found when $|k| < K$ and $\sigma < M$.

We try to find out the largest σ such that $-\sigma$ is still a negative eigenvalue of $JL(k)$. Define the set $\Omega = \{\, k \,|\, \exists -\sigma \in \sigma(JL(k)),\ -\sigma < -\delta/2\}$. One can see that Ω is a bounded non-empty open set and that $k \mapsto -\sigma(k)$ is continuous in $\bar{\Omega}$. We fix k_0, σ_0 such that

$$-\sigma_0 = -\sigma(k_0) = \inf\{-\sigma(k) |\, k \in \bar{\Omega}\} < -\delta/2 < 0.$$

Let us fix $\epsilon_0 > 0$ and an interval $I_0 \subset \Omega$ small enough with $k_0 \in I_0$ so that $-\sigma_0 \le -\sigma(k) \le -\sigma_0 + \epsilon_0$ in I_0 by the continuity of $k \mapsto -\sigma(k)$. Thanks to Lemma 2.4, we can can choose an eigenvector $V(k)$ depending analytically on k on I_0. By elliptic regularity, we have that $V(k) \in H^\infty$. Finally since one has $\sigma(k) = \sigma(-k)$ and $V(k) = V(-k)$, let us set $I = I_0 \cup -I_0$ and

$$V^0(t, x, y) = \int_I e^{-\sigma(k)t} e^{iky} V(k) dk, \qquad t \ge 0.$$

Then V^0 is real and is a solution of (2.12). We have for any $s, \alpha \in \mathbb{N}$ that

$$\|\partial_t^\alpha V^0(t, \cdot)\|_{H^s(\mathbb{R}^2)}^2 = C \int_I e^{-2\sigma(k)t} \sum_{s_1+s_2 \le s} |\sigma(k)|^{2\alpha} k^{2s_2} |\partial_x^{s_1} V(k)|_{L^2(\mathbb{R})}^2 dk.$$

and hence, there exist numbers c_{s,α,ϵ_0} such that for any $t \geq 0$

$$\|\partial_t^\alpha V^0(t,\cdot)\|_{H^s(\mathbb{R}^2)} \leq c_{s,\alpha,\epsilon_0} e^{-(\sigma_0-\epsilon_0)t}.$$

This yields similar estimates for $U^0 = PV^0$. This ends the proof of Proposition 2.5. $\square$

Proposition 2.6. *For any $M > 0$, there exists*

$$U^a = U^0 + \sum_{j=1}^{M+1} \delta^j U^j, \qquad U^j \in C^\infty(\mathbb{R}_+, H^\infty(\mathbb{R}^2)), \quad \delta \in \mathbb{R}$$

such that for every j, one has $U^j(0) = 0$ and the estimates

$$\|U^j(t)\|_{E^s} \leq C_{s,j} e^{-(j+1)(\sigma_0-\epsilon_0)t}, \qquad \forall t \geq 0$$

for ϵ_0 sufficiently small ($\epsilon_0 < \sigma_0/2$) for some numbers $C_{s,j}$ independent of t and σ_0 the eigenvalue chosen in Proposition 2.5. Moreover, define $V^a = Q + \delta U^a$. Then V^a is an approximate solution of (2.5) in the sense that

$$\partial_t V^a - \mathcal{F}(V^a) = R^{ap}$$

where R^{ap} satisfying the estimate

$$\|R^{ap}\|_{E^s} \leq C_{M,s} \delta^{M+3} e^{-(M+3)(\sigma_0-\epsilon_0)t}, \qquad t \geq 0.$$

Proof. We follow the idea of proof of [24, Proposition 6.3]. The Taylor expansion of $\mathcal{F}$ is

$$\mathcal{F}(Q + \delta U) = \mathcal{F}(Q) + \sum_{k=1}^{M+2} \frac{\delta^k}{k!} D^k \mathcal{F}[Q](U, \dots, U) + \delta^{M+3} R_{M,\delta}(U).$$

Plugging the expansion of U^a into (2.5) gives the equations for $j \geq 1$ that

$$\partial_t U^j - J\Lambda U^j = \sum_{p=2}^{j+1} \sum_{\substack{0 \leq l_1,\dots,l_p \leq j-1 \\ l_1+\dots+l_p = j+1-p}} \frac{1}{p!} D^p \mathcal{F}[Q](U^{l_1}, \dots, U^{l_p}). \tag{2.13}$$

We note the right-hand side of (2.13) as S^j. Applying Fourier transform in y to (2.13), we get that

$$\partial_t \hat{U}^j - J\Lambda(k)\hat{U}^j = \hat{S}^j, \qquad j \geq 1.$$

So we need to consider first the equation

$$\partial_t U - J\Lambda(k) U = F.$$

In order to estimate the solution, we introduce

$$|U(t)|^2_{X^s_k} := \sum_{0 \leq \alpha+\beta \leq s} \Big(|\partial_t^\alpha \partial_x^\beta U_1(t,\cdot)|^2_{H^1(\mathbb{R})} + |\partial_t^\alpha \partial_x^\beta U_2(t,\cdot)|^2_{\dot{H}^{\frac{1}{2}}_k(\mathbb{R})} \Big),$$

with

$$|\varphi|^2_{\dot{H}^{\frac{1}{2}}_k(\mathbb{R})} = \left| \frac{|D_x|}{1 + |D_x|^{\frac{1}{2}}} \varphi \right|^2_{L^2(\mathbb{R})} + |k|^2 |\varphi|^2_{L^2(\mathbb{R})}.$$

We have the following statement.

Proposition 2.7. *Fix $\gamma > \sigma_0 > 0$. Assume that $F(t,x,k)$ satisfies uniformly for $|k| \leq K$ the estimate*

$$\sum_{\alpha+\beta \leq s} \|\partial_t^\alpha \partial_x^\beta F(t,\cdot,k)\|_{L^2} \leq M_s e^{-\gamma t}, \qquad t \geq 0. \tag{2.14}$$

Then we can find a solution U of

$$\partial_t U - J\Lambda(k) U = F,$$

defined for $t \geq 0$ such that there exists constant C_s depending on M_{s+s_0} for some fixed $s_0 \geq 0$ such that uniformly for $|k| \leq K$, we have

$$|U(t,\cdot)|_{L^2} + |U(t,\cdot)|_{X^s_k} \leq C_s e^{-\gamma t}, \qquad t \geq 0.$$

Proof. Let $V = P^{-1}U$, then the equation for U is equivalent to the equation

$$\partial_t V = JL(k)V + P^{-1}F. \tag{2.15}$$

Let $\bar{\sigma} > 0$ be such that $\gamma > \bar{\sigma} > \sigma_0$. By a simple lifting argument, we get from [24, Proposition 6.4] and the symmetry of the spectrum pointed out in Lemma 2.1 that the semi-group estimate

$$\|e^{t\,JL(k)}V_0\|_{L^2\cap X^s_k} \leq C_s\, e^{\bar{\sigma}t}(|V_0|_{L^2} + |V_0|_{X^{s+s_0}_k}), \quad t \geq 0\,,$$

holds for some fixed derivative loss s_0 (we could avoid it, but this is not important in our construction). By using again the reversibility of the system, we actually obtain that

$$\|e^{t\,JL(k)}V_0\|_{L^2\cap X^s_k} \leq C_s\, e^{\bar{\sigma}|t|}(|V_0|_{L^2} + |V_0|_{X^{s+s_0}_k}), \quad \forall t \in \mathbb{R}. \tag{2.16}$$

Let us choose the solution of (2.15) given by

$$V(t) = -\int_t^{\infty} e^{(t-\tau)JL(k)} P^{-1}F(\tau)d\tau.$$

Then, thanks to (2.16), we have

$$|V(t)|_{L^2} + |V(t)|_{X^s_k} \leq C_s \int_t^{\infty} e^{\bar{\sigma}(\tau-t)} e^{-\gamma\tau} d\tau \leq C_s e^{-\gamma t},$$

with C_s having the claimed structure. This ends the proof of Propoposition 2.7. ☐

End of the proof of Proposition 2.6. We can finish the proof by induction on j as in the proof of Proposition 6.3 in [24]. Assume that we already built $(U^l)_{l\leq j-1}$ whose Fourier transforms with respect to the y variable are compactly supported in k and that satisfy uniformly for $|k| \leq K_l$ the estimates

$$|\hat{U}^l(t,k)|_{E^s} \leq C_{s,l} e^{-(l+1)(\sigma_0-\epsilon_0)t}, \quad l \leq j-1$$

where the $|\cdot|_{E^s}$ norm for functions of t and x is naturally defined as

$$|V(t)|_{E^s} = \sum_{|\alpha|\leq s} |\partial^{\alpha}_{t,x}V(t)|_{L^2(\mathbb{R})}.$$

To construct U^j, we first observe that thanks to the induction assumption we obtain as in the proof of Proposition 6.3 in [24] (hence in particular by using Proposition 3.9 in [24] to handle the terms coming from the Dirichlet-Neumann operator) that

$$\|\hat{S}^j(t,k)\|_{E^s} \leq \tilde{C}_{s,j} e^{-(j+1)(\sigma_0-\epsilon_0)t}$$

uniformly for $|k| \leq K_j$. Then we define U_j by

$$\hat{U}^j(t,k) = -P\int_t^{\infty} e^{(t-\tau)JL(k)} P^{-1}\hat{S}^j(\tau,k)d\tau.$$

Thanks to the semigroup estimate (2.16), we get since $\gamma = (j+1)(\sigma_0-\epsilon_0) > \sigma_0 > 0$ that

$$\|\hat{U}^j(t,k)\|_{E^s} \leq C_{s,j} e^{-(j+1)(\sigma_0-\epsilon_0)t}$$

and the estimate for U^j follows by using the Bessel identity and the fact that $\hat{U}^j(t,k)$ is compactly supported in k.

2.4. **Proof of Theorem 1.3.** Recall that we already defined $V^a = Q + \delta U^a$ in Proposition 2.6 as an approximate solution U of the water-waves system (2.5). In order to get a true solution of (2.5), we still need to consider a remainder U^R such that

$$U = V^a + U^R,$$

becomes an exact solution. The system for U^R is

$$\begin{cases} \partial_t U^R = \mathcal{F}(V^a + U^R) - \mathcal{F}(V^a) - R^{ap}, & t > 0, \\ U^R(0) \quad \text{to be fixed later.} \end{cases} \tag{2.17}$$

Before solving the system for U^R we need to introduce more notations. For $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$ we note $\partial^\alpha = \partial_t^{\alpha_0}\partial_x^{\alpha_1}\partial_y^{\alpha_2}$. For $U(t) = (U_1(t), U_2(t))^t$ and $k \in \mathbb{N}$ we define X^k by

$$\|U(t)\|_{X^k}^2 = \sum_{|\alpha|\le k} (\|\partial^\alpha U_1(t)\|_{H^1}^2 + \|\partial^\alpha U_2(t)\|_{H^{\frac{1}{2}}}^2)$$

and we note

$$\|U\|_{X^k_{t,T}} = \sup_{t\le\tau\le T} \|U(\tau)\|_{X^k}.$$

We define W^k and $W^k_{t,T}$ by

$$\|u(t)\|_{W^k} = \sum_{|\alpha|\le k} \|\partial^\alpha u(t)\|_{L^\infty(\mathbb{R}^2)}, \quad \|u\|_{W^k_{t,T}} = \sup_{t\le\tau\le T} \|u(\tau)\|_{W^k}.$$

We will use an approximate sequence of solutions $\{U^n\}$ for water-waves problem to prove that there exists a global-in-time solution $U^R(t)$ of (2.17). Let $\{T_n\}$ be a strictly increasing sequence such that $T_n > 0$ and $T_n \to +\infty$ as $n \to \infty$. First of all, we define $U^n(t)$ as the solution of the water-waves system

$$\begin{cases} \partial_t U^n = \mathcal{F}(V^a + U^n) - \mathcal{F}(V^a) - R^{ap}, & t \le T_n, \\ U^n(T_n) = 0. \end{cases} \tag{2.18}$$

Note that we solve the problem backward in time. For the water-waves system there is no problem to do so because of the reversibility. We have local well-posedness for (2.18) as in [24] (see also [2], [5], [26]). The first step is thus to prove that U^n is defined on the whole interval $[0, T^n]$ and that it satisfies an appropriate estimate. This will be a consequence of the following a priori estimate

Proposition 2.8. *Let $U^n(t)$ be a smooth solution of (2.18) on $[T, T_n]$ satisfying $1 - \|\eta^a(t)\|_{L^\infty} - \|\eta^n(t)\|_{L^\infty} \ge c_0 > 0$ for $t \in [T, T_n]$. Then for $m \ge 2$, $s \ge 5$ and $t \in [T, T_n]$ we have the estimate*

$$\begin{aligned} \|U^n(t)\|_{X^{m+3}}^2 \quad &\le \quad \omega(\|R^{ap}\|_{X^{m+3}_{t,T_n}} + \|V^a\|_{W^{m+s}_{t,T_n}} + \|U^n\|_{X^{m+3}_{t,T_n}}) \\ &\quad \times\Big[\|R^{ap}\|_{X^{m+3}_{t,T_n}}^2 + \int_t^{T_n} (\|U^n(\tau)\|_{X^{m+3}}^2 + \|R^{ap}(\tau)\|_{X^{m+3}}^2) d\tau\Big] \end{aligned}$$

with $\omega : \mathbb{R}^+ \to [1, +\infty]$ is a continuous increasing function.

Proposition 2.8 can be directly obtained from [24], Theorem 7.1. Indeed, let us define the isometry $\tilde{\cdot}$ by

$$\tilde{U}(\tau, \tilde{x}, \tilde{y}) = \big(U_1(T^n - \tau, -\tilde{x}, -\tilde{y}), -U_2(T^n - \tau, -\tilde{x}, -\tilde{y})\big)^t.$$

Then, we note that U^n solves (2.18) if and only if $\tilde{U}^n(\tau, \tilde{x}, \tilde{y})$ solves

$$\partial_\tau \tilde{U}^n = \mathcal{F}(\tilde{V}^a(\tau) + \tilde{U}^n(\tau)) - \mathcal{F}(\tilde{V}^a(\tau)) + \tilde{R}^{ap}(\tau), \quad \tau \in [0, T^n]$$

with the initial data $\tilde{U}^n(0) = 0$.

We can now convert the a priori estimate of Proposition 2.8 into the following existence result.

Proposition 2.9. *Let $m \geq 2$. Then there exists M large enough in the definition of V^a and δ sufficiently small such that the following holds true. For every $T_n > 0$ there exists a unique solution of (2.18) defined on $[0, T^n]$ and satisfying*

$$1 - \|\eta^a\|_{L^\infty} - \|\eta^n(t)\|_{L^\infty} > 0, \quad \|U^n(t)\|_{X^{m+3}} \leq C_{M,m}\delta^{M+3}e^{-(M+3)(\sigma_0-\epsilon_0)t}, \qquad t \in [0, T_n].$$

Proof. From [24, Section 8] for example and the above reversibility argument, we know the local-in-time existence for the solution $U^n(t)$ of system (2.18) going backwards from time T_n. We want to prove that this solution exists on the whole time interval $[0, T_n]$. By Proposition 2.6 and Proposition 2.8, we have that, at least for t close to T_n

$$\|U^n(t)\|^2_{X^{m+3}} \leq \omega(C + \|U^n\|_{X^{m+3}_{t,T_n}} + C_{M,m}\delta) \times \Big(\int_t^{T_n} \|U^n(\tau)\|^2_{X^{m+3}} d\tau + C_{M,m}\delta^{2(M+3)}e^{-2(M+3)(\sigma_0-\epsilon_0)t} \Big). \tag{2.19}$$

Define

$$T^* = \inf\{T \in [0, T_n] \,:\, \forall t \in [T, T_n],\, \|U^n(t)\|_{X^{m+3}} \leq 1,\, 1 - \|\eta^a\|_{L^\infty} - \|\eta^n(t)\|_{L^\infty} \geq c_0 > 0\}.$$

We deduce from (2.19) that

$$\|U^n(t)\|^2_{X^{m+3}} \leq \omega(C + C_{M,m}\delta)\Big(\int_t^{T_n} \|U^n(\tau)\|^2_{X^{m+3}} d\tau + C_{M,m}\delta^{2(M+3)}e^{-2(M+3)(\sigma_0-\epsilon_0)t} \Big)$$

for $t \in [T^*, T_n]$. Now we shall fix the value of M and impose a smallness condition on δ. Let M and δ be such that

$$2(M+3)(\sigma_0 - \epsilon_0) > \omega(C + C_{M,m}\delta)$$

Such a choice is possible thanks to the continuity of ω. Indeed we can first take M large enough so that $2(M+3)(\sigma_0 - \epsilon_0) > \omega(C) + 2$ and then δ small enough so that $|\omega(C + C_{M,m}\delta) - \omega(C)| < 1$. Therefore, we arrive at the bound

$$\frac{d}{dt}\Big(- e^{\omega(C+C_{M,m}\delta)t} \int_t^{T_n} \|U^n(\tau)\|^2_{X^{m+3}} d\tau \Big) \leq C_{M,m}\delta^{2(M+3)}e^{\omega(C+C_{M,m}\delta)t - 2(M+3)(\sigma_0-\epsilon_0)t}.$$

Integrating on both sides with respect to time from t to T_n leads to

$$\int_t^{T_n} \|U^n(\tau)\|^2_{X^{m+3}} d\tau \leq C_{M,m}\delta^{2(M+3)}e^{-2(M+3)(\sigma_0-\epsilon_0)t},$$

which gives

$$\|U^n(t)\|^2_{X^{m+3}} \leq C_{M,m}\delta^{2(M+3)}e^{-2(M+3)(\sigma_0-\epsilon_0)t} \tag{2.20}$$

for any $t \in [T^*, T_n]$. Now we let δ small enough such that $C_{M,m}\delta^{2(M+3)} < 1$ and $1 - \|\eta_\varepsilon\|_{L^\infty} - C_{M,m}\delta > c_0$. Then by definition, we have $T^* \leq 0$, hence the solution $U^n(t)$ for (2.18) can be extended to the whole time interval $[0, T_n]$ with the claimed estimates. □

We can now finish the proof of Theorem 1.3 by invoking some standard compactness arguments.

Step 1. Existence of the global solution $U(t)$. Thanks to Proposition 2.8 and Proposition 2.9, we get the approximation sequence $\{U^n\}$ of solutions of (2.18) which satisfy

$$\|U^n(t)\|_{X^{m+3}} \leq C_{M,m}\delta^{M+3}e^{-(M+3)(\sigma_0-\epsilon_0)t}, \qquad \forall t \in [0, T_n].$$

Let $\psi \in C_0^\infty(-1/2, 1/2)$ be a bump function such that $\psi(x) = 1$ for $x \in (-1/4, 1/4)$. We extend $U^n(t)$ as zero for $t > T_n$ and we set

$$\tilde{U}^n(t) = \psi(t/T_n)U^n(t), \quad t \geq 0.$$

Then, we have

$$\partial_t \tilde{U}^n(t) = \frac{1}{T_n}\psi'(t/T_n)U^n(t) + \psi(t/T_n)\partial_t U^n(t), \quad t > 0$$

and thus new sequence $\{\tilde{U}^n(t)\}$ satisfies

$$\|\tilde{U}^n(t)\|_{H^{m+4}\times H^{m+7/2}} \leq C_{M,m}\delta^{M+3}e^{-(M+3)(\sigma_0-\epsilon_0)t}, \quad t \geq 0$$

and

$$\|\partial_t \tilde{U}^n(t)\|_{H^{m+3}\times H^{m+5/2}} \leq C_{M,m}\delta^{M+3}e^{-(M+3)(\sigma_0-\epsilon_0)t}, \qquad \forall t \geq 0.$$

By a standard compactness argument, we obtain that there exists a subsequence $\{\tilde{U}^{n_k}(t)\}$ and

$$U^R(t) \in L^\infty([0,\infty),\ H^{m+4}\times H^{m+7/2})$$

such that

$$\tilde{U}^{n_k} \to U^R \quad \text{in} \quad C_{loc}(\mathbb{R}_+, H^{m+3}_{loc}\times H^{m+5/2}_{loc}) \quad \text{as} \quad n_k \to \infty$$

and

$$\|U^R(t)\|_{H^{m+4}\times H^{m+7/2}} \leq C_{M,m}\delta^{M+3}e^{-(M+3)(\sigma_0-\epsilon_0)t} \qquad \forall t \in [0,\infty).$$

Moreover $U^R(t)$ is the solution of (2.17) since $U^{n_k}(t)$ is a solution of (2.17) on $(0, T_{n_k}/4)$. Going back to water-waves system (2.5) we deduce that there is a global solution $U(t) = V^a(t) + U^R(t)$ for system (2.5).

Step 2. Behavior when $t \to +\infty$. By the definition of V^a we have

$$U(t) = V^a(t) + U^R(t) = Q + \sum_{j=0}^{M+1}\delta^{j+1}U^j(t) + U^R(t),$$

and from Proposition 2.6 and (2.20) we know that for any $t \in [0,\infty)$

$$\|U^j(t)\|_{E^s} \leq C_{M,s}e^{-(j+1)(\sigma_0-\epsilon_0)t},$$
$$\|U^R(t)\|_{X^{s+3}} \leq C_{M,s}\delta^{M+3}e^{-(M+3)(\sigma_0-\epsilon_0)t}$$

with $s \geq 2$. This yields

$$\|U^j(t)\|_{H^s} \leq C_{M,s}e^{-(j+1)(\sigma_0-\epsilon_0)t}, \quad \|U^R(t)\|_{H^s} \leq C_{M,s}e^{-(M+3)(\sigma_0-\epsilon_0)t}, \qquad \forall t \in [0,\infty)$$

which shows that

$$\lim_{t\to+\infty}\|U(t) - Q(x)\|_{H^s} = 0.$$

Step 3. It remains to check the condition (1.9) for U. Since

$$U(t) = V^a(t) + U^R(t) = Q(x) + \delta U^a(t) + U^R(t),$$

we have at $t = 0$ that

$$U(0) = Q(x) + \delta\sum_{j=0}^{M+1}\delta^j U^j(0) + U^R(0)$$

and

$$\partial_y(U(0)) = \delta\partial_y(U^0(0)) + \sum_{j=1}^{M+1}\delta^{j+1}\partial_y(U^j(0)) + \partial_y(U^R(0)).$$

We know from the definition of U^0 that $\partial_y(U^0(0)) \neq 0$. We can use Proposition 2.6 and step 1 to get that

$$\|U^j(0)\|_{E^s} \leq C_{s,j}, \quad \|U^R(0)\|_{X^{m+3}} \leq C_{M,m}\delta^{M+3}.$$

This yields that $\partial_y(U(0)) \neq 0$ for δ possibly smaller. This ends the proof of Theorem 1.3. $\square$

3. Preliminaries for the proof of Theorem 1.2

3.1. **The Dirichlet-Neumann operator.** Let us recall that the Dirichlet-Neumann operator $G[\eta]$ is defined by

$$G[\eta]\psi = \sqrt{1+|\partial_x\eta|^2}\partial_n\varphi|_{z=\eta}$$

where n is the unit outward normal vector and φ solves

$$\begin{cases} \Delta\varphi = 0, \quad \text{in} \quad -H < z < \eta(x) \\ \varphi|_{z=\eta} = \psi, \quad \partial_n\varphi|_{z=-H} = 0 \end{cases}$$

We can rewrite the elliptic problem on $-H < z < \eta(x)$ as an equivalent problem on a flat strip $\mathcal{S} = \mathbb{R}\times[-1,0]$:

$$\begin{cases} \nabla_{x,z}\cdot P\nabla_{x,z}\tilde{\varphi} = 0, \quad \text{in} \quad \mathcal{S} \\ \tilde{\varphi}|_{z=0} = \psi, \quad \partial_z\tilde{\varphi}|_{z=-1} = 0 \end{cases} \tag{3.1}$$

where $\tilde{\varphi}(x,z) = \varphi(x, Hz + (z+1)\eta)$ and

$$P = \begin{pmatrix} H+\eta & -(z+1)\partial_x\eta \\ -(z+1)\partial_x\eta & \frac{1+(z+1)^2(\partial_x\eta)^2}{H+\eta} \end{pmatrix}.$$

With the notation $\partial_n^P = (0,1)^t\cdot P\,\nabla_{x,z}$, one can see that the D-N operator associated to the above problem can be written as

$$G[\eta]\psi = \partial_n^P\tilde{\varphi}|_{z=0} = -\partial_x\eta\partial_x\tilde{\varphi}|_{z=0} + \frac{1+(\partial_x\eta)^2}{H+\eta}\partial_z\tilde{\varphi}|_{z=0}. \tag{3.2}$$

Let us recall the following properties:

Proposition 3.1. *For $\eta\in H^\infty(\mathbb{R})$ with $H+\eta\geq c_0>0$, we have*

(1) *$G[\eta]$ is symmetric on $L^2(\mathbb{R})$:*

$$(G[\eta]u, v) = (u, G[\eta], v), \quad \forall\, u, v\in H^{\frac{1}{2}}(\mathbb{R})$$

(2) *There exists $c>0$, $C>0$ such that for every $u\in H^{\frac{1}{2}}(\mathbb{R})$*

$$|(G[\eta]u, v)| \leq C|\mathfrak{P}u|_{L^2}|\mathfrak{P}v|_{L^2}, \forall u,\, v\in H^{\frac{1}{2}}(\mathbb{R}) \tag{3.3}$$

$$(G[\eta]u, u) \geq c|\mathfrak{P}u|_{L^2}^2,\, \forall u\in H^{\frac{1}{2}}(\mathbb{R}) \tag{3.4}$$

where $\mathfrak{P}$ is the Fourier multiplier

$$\mathfrak{P} = (1-\partial_x^2)^{-\frac{1}{4}}\partial_x.$$

(3) *the linear operator $G[\eta]: H^{s+1}(\mathbb{R})\to H^s(\mathbb{R})$ is continuous for every $s\in\mathbb{R}$*

(4) *We have the following commutator estimates*

$$|[\partial_x^s, G[\eta]]u|_{H^{\frac{1}{2}}} \leq C_s|\mathfrak{P}u|_{H^s}, \quad \forall u\in H^{s+\frac{1}{2}}(\mathbb{R}),$$

$$|(f\partial_x u, G[\eta]u)| \leq C_f|\mathfrak{P}u|_{L^2}, \quad \forall u\in H^{\frac{1}{2}}(\mathbb{R}), \forall f\in H^\infty(\mathbb{R}).$$

(5) *We have the explicit expression for the derivative of the Dirichlet-Neumann operator with respect to η:*

$$D_\eta(G[\eta]u)\cdot\zeta = -G[\eta](\zeta Z) - \partial_x(v\zeta)$$

with

$$Z = Z[\eta, u] = \frac{G[\eta]u + \partial_x\eta\partial_x u}{1+|\partial_x\eta|^2},\, v = v[\eta, u] = \partial_x u - Z\partial_x\eta.$$

(6) *Finally, for $s > 1/2$, we have that*

$$\left|D_\eta^j(G[\eta]u)\cdot(h_1,\cdots,h_j)\right|_{H^{s-\frac{1}{2}}} \leq C_s\left|\mathfrak{P}u\right|_{H^s}\prod_{i=1}^{j}|h_i|_{H^{s+1}}.$$

For the proof of these statements we refer to [17] or [25] section 3. We have assumed that η is smooth since we shall use this proposition when η is a solitary wave. Most of the estimates are actually still true when η only has limited Sobolev regularity.

In order to perform our construction of multisolitons, we shall also need results about the action of the Dirichlet-Neumann operator on localized functions.

Proposition 3.2. *Assume that $\psi \in C_b^\infty(\mathbb{R})$ has an exponential decay:*

$$\exists\, d>0,\ \forall\,\alpha\in\mathbb{N},\alpha\geq 1,\exists\, C_\alpha,\forall\, x\in\mathbb{R},|\partial_x^\alpha\psi(x)|\leq C_\alpha e^{-d|x|}. \tag{3.5}$$

Then for $\eta \in H^\infty(\mathbb{R})$ with $H+\eta \geq c_0 > 0$, $G[\eta]\psi$ also has an exponential decay, that is, for any $\alpha \in \mathbb{N}$, there exist a constant C_α depending on α and $0 < \epsilon < d$ independent of α such that for every $x \in \mathbb{R}$,

$$\left|\partial_x^\alpha\big(G[\eta]\psi\big)(x)\right| \leq C_\alpha e^{-\epsilon|x|}.$$

Remark 3.3. *We only assume that the derivatives of the function ψ are exponential decaying while the function itself is only bounded as it is the case for the solitary waves (see (4.2) below). Note that this still yields that $G[\eta]\psi$ is exponentially decaying. The heuristic reason is that the Dirichlet-Neumann operator behaves like a derivative. Again, in Proposition 3.2, we are not interested in the way the estimates depend on the regularity of the surface since we will use it for solitary waves.*

The result of Proposition 3.2 uses in an essential way that the fluid domain is bounded in the z direction (we use a Poincaré inequality). The statement of Proposition 3.2 does not hold in the case of an infinite bottom because of a singularity at the low frequencies which affects the propagation of the exponential decay.

Proof of Proposition 3.2. We use as in [24] the decomposition

$$\tilde{\varphi}(x,z) = \varphi_0(x,z) + u(x,z) \tag{3.6}$$

with φ_0 that solves the elliptic problem

$$-\Delta_{x,z}\varphi_0 = 0,\ (x,z)\in\mathcal{S}, \qquad \varphi_0(x,0)=\psi,\ \partial_z\varphi_0(x,-1)=0.$$

This allows to transform the elliptic problem (3.1) for φ into an elliptic problem with homogeneous boundary conditions for u:

$$\begin{cases} -\nabla_{x,z}\cdot(P\nabla_{x,z}u) = \nabla_{x,z}\cdot(P\nabla_{x,z}\varphi_0), & \text{in} \quad \mathcal{S} \\ u|_{z=0}=0, \quad \partial_z u|_{z=-1}=0 \end{cases} \tag{3.7}$$

At first, we shall study the decay properties of φ_0. We first observe that φ_0 is given by the explicit formula

$$\widehat{\varphi_0}(\xi,z) = \frac{\cosh\big(\xi(z+1)\big)}{\cosh(\xi)}\hat{\psi}(\xi) \tag{3.8}$$

where $\hat{\cdot}$ stands for the Fourier transform in the x variable. From this expression, we get in particular that

$$\mathcal{F}_x(\partial_x\varphi_0)(\xi,z) = \frac{\cosh\big(\xi(z+1)\big)}{\cosh(\xi)}\mathcal{F}_x(\partial_x\psi)(\xi).$$

The exponential decay will follow from Paley-Wiener type arguments. Since $\partial_x\psi$ and its derivatives have exponential decay, we get that $\mathcal{F}_x(\partial_x\psi)$ has an holomorphic extension to $|\mathrm{Im}\,\xi|<d$ and by integration by parts that it satisfies for every $\delta\in(0,d)$ the estimate

$$\big|\mathcal{F}_x(\partial_x\varphi_0)(\xi,z)\big|\leq\frac{C_N}{1+|\xi|^N},\quad\forall\xi,\,z,\,|\mathrm{Im}\ \xi|\leq\delta,\,z\in[-1,0]$$

for every $N\in\mathbb{N}$. Since $\xi\mapsto\frac{\cosh\big(\xi(z+1)\big)}{\cosh(\xi)}$ has an holomorphic bounded (uniformly in z) extension to $|\mathrm{Im}\ \xi|\leq\delta$ for any $\delta\in(0,\pi/2)$, we can use contour deformation to write that

$$\partial_x\varphi_0(x,z)=\int_{\mathbb{R}}e^{ix\xi}\frac{\cosh\big(\xi(z+1)\big)}{\cosh(\xi)}\mathcal{F}_x(\partial_x\psi)(\xi)\,d\xi=\int_{Im\,\xi=\delta}e^{ix\xi}\frac{\cosh\big(\xi(z+1)\big)}{\cosh(\xi)}\mathcal{F}_x(\partial_x\psi)(\xi)\,d\xi$$

for any δ such that $|\delta|<\mathrm{Min}\ (d,\pi/2)$. This yields by choosing $\delta=2\epsilon\,\mathrm{sign}\ x$, with ϵ sufficiently small the estimate

$$|\partial_x\varphi_0(x,z)|\lesssim e^{-2\epsilon|x|},\ \forall(x,z)\in\mathcal{S}.$$

In a similar way, we get from (3.8) that

$$\partial_z\hat{\varphi}_0(x,z)=\frac{\sinh\big(\xi(z+1)\big)}{\cosh\xi}\xi\hat{\psi}(\xi)=\frac{\sinh\big(\xi(z+1)\big)}{\cosh\xi}\frac{1}{i}\mathcal{F}_x(\partial_x\psi)(\xi)$$

and hence, since $\partial_x\psi$ and its derivatives have exponential decay, the same arguments as above yield the estimate

$$|\partial_z\varphi_0(x,z)|\lesssim e^{-2\epsilon|x|},\ \forall(x,z)\in\mathcal{S}.$$

The estimates for higher order derivatives can also be obtained from the same arguments, we find in the end:

$$\forall\beta,\,|\beta|\geq1,\quad|\partial_{x,z}^{\beta}\varphi_0(x,z)|\lesssim C_\beta e^{-2\epsilon|x|},\quad\forall(x,z)\in\mathcal{S}.\tag{3.9}$$

It remains to estimate the solution u of (3.7). Let us define $v(x,z)$ such that

$$u(x,z)=e^{-\epsilon\langle x\rangle}v(x,z),\quad\langle x\rangle:=(1+x^2)^{\frac{1}{2}}$$

with $\epsilon>0$ small enough and to be fixed later. One has the elliptic problem for $v(x,z)$

$$\begin{cases}-\nabla_{x,z}\cdot(P\nabla_{x,z}v)-[\nabla_{x,z}\cdot P\nabla_{x,z},\,e^{\epsilon\langle x\rangle}]e^{-\epsilon\langle x\rangle}v=e^{\epsilon\langle x\rangle}\nabla_{x,z}\cdot(P\nabla_{x,z}\varphi_0),\quad\text{in}\quad\mathcal{S}\\ v|_{z=0}=0,\quad\partial_z v|_{z=-1}=0\end{cases}\tag{3.10}$$

We shall first estimate Sobolev norms of $v(x,z)$.

1) Lower-order elliptic estimate for v. By integration by parts, we get that

$$\begin{aligned}&\int_{\mathcal{S}}P\nabla_{x,z}v\cdot\nabla_{x,z}v dxdz\\&=\int_{\mathcal{S}}\Big([\nabla_{x,z}\cdot P\nabla_{x,z},\,e^{\epsilon\langle x\rangle}]e^{-\epsilon\langle x\rangle}v\Big)\,vdxdz+\int_{\mathcal{S}}e^{\epsilon\langle x\rangle}\nabla_{x,z}\cdot(P\nabla_{x,z}\varphi_0)\,v\,dxdz\\&:=A_1+A_2.\end{aligned}$$

For the left hand side, we have by the assumption on η that

$$\int_{\mathcal{S}}P\nabla_{x,z}v\cdot\nabla_{x,z}v dxdz\geq c\|\nabla_{x,z}v\|^2$$

for some $c>0$ independent of ϵ. Here and in the sequel the norm $\|\cdot\|$ is the $L^2(\mathcal{S})$ norm. Next, we can compute

$$[\nabla_{x,z}\cdot P\nabla_{x,z},\,e^{\epsilon\langle x\rangle}]=(H+\eta)[\partial_x^2,e^{\epsilon\langle x\rangle}]-2(z+1)\partial_x\eta[\partial_x,e^{\epsilon\langle x\rangle}]\partial_z$$

which yields

$$|A_1|\leq C\big(\epsilon\|\nabla_{x,z}v\|+\epsilon^2\|v\|\big)\|v\|.$$

For the second term in the right hand side, we can use (3.9) to get

$$|A_2| \leq C_\epsilon \|v\| \leq \epsilon^2 \|v\|^2 + C_\epsilon,$$

where the last inequality comes from the Young inequality, for some harmless number C_ϵ (that depends on ϵ). Summing all these estimates up one gets that

$$\|\nabla_{x,z} v\|^2 \leq C(\epsilon^2 \|v\|^2 + \epsilon \|\nabla_{x,z} v\| \|v\| + C_\epsilon)$$

To conclude, we note that since $v|_{z=0} = 0$ and $\mathcal{S}$ is bounded in the z direction, we have the Poincaré inequality:

$$\|v\| \leq C \|\nabla_{x,z} v\|.$$

Plugging this into the above estimate yields

$$\|\nabla_{x,z} v\| \leq C, \quad \|v\| \leq C.$$

by taking ϵ sufficiently small.
2) Higher-order estimates for v. These estimates will follow from an induction argument and standard elliptic regularity theory. Indeed, we can write the elliptic problem (3.10) under the form

$$-\nabla_{x,z} \cdot \big(P \nabla_{x,z} v\big) = F, \quad (x,z) \in \mathcal{S}, \quad v(x,0) = 0, \, \partial_z v(x,-1) = 0.$$

From standard elliptic regularity theory, we have that for $s \geq 0$

$$\|v\|_{H^{s+2}(\mathcal{S})} \leq C\big(\|v\|_{H^1(\mathcal{S})} + \|F\|_{H^s(\mathcal{S})}\big).$$

By using the estimate (3.9) and a standard commutator estimate, we get

$$\|F\|_{H^s(\mathcal{S})} \leq C\big(1 + \|v\|_{H^{s+1}(\mathcal{S})}\big).$$

Consequently starting from the H^1 estimate that we have already proven, we get by induction that for every $s \geq 0$, $\|v\|_{H^s(\mathcal{S})} \leq C_s$. By Sobolev embedding, we thus obtain that $e^{\epsilon(1+x^2)^{\frac{1}{2}}} u$ and all its derivatives are bounded in $\mathcal{S}$ that is to say:

$$\forall \beta, \, |\partial_{x,z}^\beta u(x,z)| \leq C_\beta e^{-\epsilon|x|}, \quad \forall (x,z) \in \mathcal{S}. \tag{3.11}$$

To end the proof of Proposition 3.2, it suffices to combine the expression (3.2) of the Dirichlet-Neumann operator (note that it always involves a derivative applied to φ) with the decomposition (3.6) and the estimates (3.9), (3.11). This ends the proof of Proposition 3.2. □

We shall also use the following corollary.

Corollary 3.4. *Let $\psi \in C_b^\infty(\mathbb{R})$ with an exponential decay property as in Proposition 3.2. Then for $\eta \in H^\infty(\mathbb{R})$ such that $H + \eta \geq c_0 > 0$ and every $a \in R$, $G[\eta](\psi(x-a))$ also has an exponential decay, i.e. there exists constant $0 < \epsilon < d$ independent of a and α such that*

$$|\partial_x^\alpha G[\eta](\psi(\cdot - a))(x)| \leq C_\alpha e^{-\epsilon|x-a|}.$$

Proof. Let us introduce the translation operator $(\tau_a f)(x) := f(x-a)$. The result follows by observing that $G[\eta]\big(\tau_a \psi\big) = \tau_a\big(G[\tau_{-a}\eta]\psi\big)$ and by using Proposition 3.2. □

3.2. **Linear stability properties of the solitary wave.** In this section, we study the linearization of the water-waves system about a solitary wave $Q_c(x-ct)$ given by Theorem 1.1. The main results of this section (in particular Proposition 3.6) are essentially contained in [21]. For the sake of completeness, we shall give proofs that use a slightly different framework which is more adapted to our purpose.

It is convenient here to go into the moving frame by changing x into $x-ct$. As in section 2.1, the linearized equation reads

$$\partial_t U = J\Lambda_c U$$

with

$$\Lambda_c = \begin{pmatrix} -P_c + g + Z_c G[\eta_\epsilon](Z_c\cdot) + Z_c\partial_x v_c & (v_c - c)\partial_x - Z_c G[\eta_c] \\ -\partial_x((v_c-c)\cdot) - G[\eta_c](Z_c\cdot) & G[\eta_c] \end{pmatrix}$$

and $Z_c = Z[Q_c]$, $v_c = v[Q_c]$ are defined in (5) Proposition 3.1. The operator P_c is the second order elliptic operator defined by

$$P_c = b\partial_x\Big(\frac{\partial_x\cdot}{\big(1+(\partial_x\eta_c)^2\big)^{\frac{3}{2}}}\Big).$$

We can also get a simpler form of the linearized system by the change of unknowns

$$V = R_c U, \quad R_c = \begin{pmatrix} 1 & 0 \\ -Z_c & 1 \end{pmatrix} \tag{3.12}$$

which yields

$$\partial_t V = J L_c V, \tag{3.13}$$

with a symmetric operator L_c defined by

$$L_c = \begin{pmatrix} -P_c + g + (v_c-c)\partial_x Z_c & (v_c-c)\partial_x \\ -\partial_x((v_c-c)\cdot) & G[\eta_c] \end{pmatrix}.$$

The two operators are related by the property

$$L_c = (R_c^{-1})^t \Lambda_c R_c^{-1}. \tag{3.14}$$

Note that in this section, we see c as a more natural parameter than ε for the solitary wave and the objects depending on it since we have not written the system in a non-dimensional form, but that L_c is conjugated to the operator $L(0)$ studied previously via the scaling transformation (2.3).

Thanks to Proposition 3.1, the quadratic form associated to L_c is naturally defined on the space $X^0 = H^1(\mathbb{R}) \times \dot{H}_*^{\frac{1}{2}}(\mathbb{R})$ where $\dot{H}_*^{\frac{1}{2}}(\mathbb{R})$ is a modified homogeneous Sobolev space defined by

$$\dot{H}_*^{\frac{1}{2}}(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}),\, \mathfrak{P}u \in L^2(\mathbb{R})\}.$$

Remark 3.5. *Note that with our definition of* $\mathfrak{P}$, *we have that* $\dot{H}_*^{\frac{1}{2}}(\mathbb{R}) \subset L^2_{loc}(\mathbb{R})$. *Indeed if* $u \in \dot{H}_*^{\frac{1}{2}}(\mathbb{R})$, *then* $v = \mathfrak{P}u \in L^2$. *Let us choose* $\chi(\xi)$ *a smooth compactly supported function such that* $\chi = 1$ *in the vicinity of zero. Then since*

$$(1-\chi(D))u = \mathfrak{P}^{-1}(1-\chi(D))v, \tag{3.15}$$

and that $\frac{1-\chi(\xi)}{\mathfrak{P}(\xi)}$ *is bounded, we get that* $(1-\chi(D))u \in L^2(\mathbb{R})$. *Next we note that*

$$\chi u = C + \int_0^x (1-\partial_x^2)^{\frac{1}{4}}(\chi v)\, dy \tag{3.16}$$

for some constant C. *Since* $(1-\partial_x^2)^{\frac{1}{4}}(\chi v) \in L^2(\mathbb{R})$, *this yields that* $\chi(D)u \in L^2_{loc}(\mathbb{R})$ *and hence that*

$$u = \chi u + (1-\chi)u \in L^2_{loc}(\mathbb{R}). \tag{3.17}$$

On $\dot{H}^{\frac{1}{2}}_*(\mathbb{R})$, we shall use the semi-norm $|u|_{\dot{H}^{\frac{1}{2}}_*} = |\mathfrak{P}u|_{L^2}$ and hence on X^0, we set

$$|U|_{X^0} = |U_1|_{H^1} + |U_2|_{\dot{H}^{\frac{1}{2}}_*}.$$

We could make X^0 a Banach space by taking the quotient by the linear space

$$0_{X^0} = \{(0,\lambda),\, \lambda \in \mathbb{R}\}. \tag{3.18}$$

Nevertheless, we refrain from doing it since it is very convenient for us to have that X^0 is a subspace of L^2_{loc} (or $\mathcal{S}'$).

Proposition 3.6. *Suppose that c satisfies* (1.7)*. There exists ε^* such that for every $\varepsilon \in (0, \varepsilon^*]$, we have for $Q_c = (\eta_c, \varphi_c)$ the corresponding solitary wave of speed c that there exists $C > 0$ such that for every $U = (U_1, U_2) \in X^0$ such that $(U, JR_c\partial_x Q_c) = (U_1, \partial_x \eta_c) = 0$,*

$$(L_c U, U) \geq C^{-1}|U|^2_{X^0}.$$

The orthogonality condition that appears in this proposition is with respect to $JR_c\partial_x Q_c$ because of the relation (3.14) since we are dealing with the operator L_c. If we translate the result into a positivity property in terms of Λ_c, we recover the natural condition $(J\partial_x Q_c, U) = 0$.

Remark 3.7. *A more natural orthogonality condition related to the general Grillakis-Shatah-Strauss framework* [15] *would be to show that if*

$$(U, JR_c\partial_x Q_c) = (U, R_c\partial_x Q_c) = 0 \tag{3.19}$$

then

$$(L_c U, U) \geq C^{-1}|U|^2_{X^0}. \tag{3.20}$$

Nevertheless, note that the condition $(U, R_c\partial_x Q_c) = 0$ is not really well defined on X^0 because the scalar product $(U_2, \partial_x \phi_c)$ would not be uniquely defined in the quotient since φ_c has different limits at $\pm\infty$. Using this remark differently, we observe that for U under the form

$$U = \lambda_1 R_c \partial_x Q_c + \lambda_2 (0,1)^t,$$

for every λ_1, one can find some λ_2 such that $(U, JR_c\partial_x Q_c) = (U, R_c\partial_x Q_c) = 0$. This yields that we can always find some U such that $L_c U = 0$ and that satisfies the orthogonality conditions (3.19) *but that $U \notin 0_{X^0}$ since one can always chose $\lambda_1 \neq 0$. Therefore,* (3.20) *under* (3.19) *is false.*

Proof. We first prove the following weaker version of the statement.

Lemma 3.8. *Let $U = (U_1, U_2) \in X^0$, $U \notin 0_{X^0}$ and such that*

$$(U, JR_c\partial_x Q_c) = (U_1, \partial_x \eta_c) = 0.$$

Then $(L_c U, U) > 0$.

Proof of Lemma 3.8. Lemma 3.8 is a variation on [24, Proposition 4.8] with different orthogonality conditions. It turns out that the orthogonality conditions used in [24, Proposition 4.8] are not appropriate for our purpose here. Using the rescaling (2.3), we already know thanks to [24, Proposition 4.8] that:

Proposition 3.9. *There exists a co-dimension two subspace X^0_1 of X^0 and a constant c_0 such that for every $U \in X^0_1$,*

$$(L_c U, U) \geq c_0 |U|^2_{X^0}.$$

As a consequence, thanks to (3.14) *and using that $R^t_c X^0 \subset X^0$, we obtain that there exists a co-dimension two subspace X^0_2 of X^0 and a constant $\tilde{c}_0$ such that for every $U \in X^0_2$,*

$$(\Lambda_c U, U) \geq \tilde{c}_0 |U|^2_{X^0}.$$

We shall now prove Lemma 3.8 by using Proposition 3.9. Note that because of (3.14) and using that $(R_c^{-1})^t J = JR_c$, it is equivalent to prove that for every $U \notin 0_{X^0}$ and such that $(U, J\partial_x Q_c) = (U_1, \partial_x \eta_c) = 0$, one has $(\Lambda_c U, U) > 0$. Recall that the solitary wave Q_c is a critical point of the Hamiltonian $H(\eta, \varphi)$, defined by

$$H(\eta, \varphi) = \frac{1}{2} \int_{-\infty}^{\infty} \Big(G[\eta]\varphi\varphi + g\eta^2 + 2b(\sqrt{1 + (\partial_x \eta)^2} - 1) - 2c\eta\partial_x\varphi \Big).$$

Thus $\nabla H(Q_c) = 0$. Differentiating the last relation with respect to c and x respectively gives

$$\Lambda_c \partial_c Q_c = J\partial_x Q_c, \quad \Lambda_c \partial_x Q_c = 0. \tag{3.21}$$

By contradiction, let us assume that there exists $y = (y_1, y_2) \in X^0$, $y \notin 0_{X^0}$ and satisfying

$$(y, J\partial_x Q_c) = (y_1, \partial_x \eta_c) = 0, \quad (\Lambda_c y, y) \le 0\,.$$

Set $Y = \operatorname{span}(y, \partial_x Q_c, \partial_c Q_c)$. We claim that $\dim Y = 3$. In order to prove this claim, we can use the following lemma.

Lemma 3.10. *Under the assumptions of Proposition 3.6, $\partial_x Q_c$ and $\partial_c Q_c$ are not co-linear and*

$$(\partial_c Q_c, J\partial_x Q_c) < 0\,. \tag{3.22}$$

We will give the proof of Lemma 3.10 later. Let us use it to prove that the vector space Y is three-dimensional. If we suppose that

$$y = \alpha\partial_x Q_c + \beta\partial_c Q_c, \quad \alpha, \beta \in \mathbb{R}$$

then by taking the scalar product (the distributional duality) with $J\partial_x Q_c$ and $(\partial_x \eta_c, 0)^t$, we get

$$0 = (y, J\partial_x Q_c) = \beta(\partial_c Q_c, J\partial_x Q_c), \quad 0 = (y_1, \partial_x \eta_c) = \alpha\|\partial_x \eta_c\|_{L^2}^2 + \beta(\partial_c \eta_c, \partial_x \eta_c). \tag{3.23}$$

Using Lemma 3.10 and (3.23), we obtain that $\alpha = \beta = 0$. Therefore $\dim Y = 3$.

Now, a similar argument shows that $Y \cap 0_{X^0} = \{(0,0)^t\}$. Indeed, it suffices to use that $(0,1)^t$ is orthogonal to $J\partial_x Q_c$ and $(\partial_x \eta_c, 0)^t$ and to take the scalar product with these vectors in a relation of type

$$\lambda_1 y + \lambda_2 \partial_x Q_c + \lambda_3 \partial_c Q_c = (0,1)^t, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

Next, for $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$, we can write by invoking Lemma 3.10 and (3.21),

$$\big(\Lambda_c(\mu_1 y + \mu_2 \partial_x Q_c + \mu_3 \partial_c Q_c), \mu_1 y + \mu_2 \partial_x Q_c + \mu_3 \partial_c Q_c\big) = \mu_1^2 (\Lambda_c y, y) + \mu_3^2 (\partial_c Q_c, J\partial_x Q_c) \le 0\,.$$

Therefore $(\Lambda_c U, U) \le 0$ for every $U \in Y$. But since $\dim Y = 3$ there exists a nontrivial $z \in Y$ such that $z \in X_2^0$ where X_2^0 is the space involved in the statement of Proposition 3.9. Therefore $|z|_{X^0} = 0$. This implies that $z \in 0_{X^0}$. But $z \in Y$ and $Y \cap 0_{X^0} = \{(0,0)^t\}$ which implies $z = 0$. Contradiction. This ends the proof of Lemma 3.8. $\square$

Proof of Lemma 3.10. Thanks to [3] (see also Theorem 1.1), we have by setting $X = x/H$

$$\frac{\eta_c}{H}(x) = -\varepsilon^2 \mathrm{ch}^{-2}\Big(\frac{\varepsilon X}{2(\beta - 1/3)^{\frac{1}{2}}}\Big) + \mathcal{O}(\varepsilon^4 e^{-c\varepsilon|X|}) \tag{3.24}$$

and

$$\frac{\varphi_c}{c\,H}(x) = -2\varepsilon(\beta - 1/3)^{\frac{1}{2}} \mathrm{th}^{-2}\Big(\frac{\varepsilon X}{2(\beta - 1/3)^{\frac{1}{2}}}\Big) + \mathcal{O}(\varepsilon^3), \tag{3.25}$$

where $\beta = \frac{b}{Hc^2}$ and $\varepsilon = \sqrt{\frac{gH}{c^2} - 1}$. Then $\partial_c \varepsilon < 0$ and using that

$$(\partial_c Q_c, J\partial_x Q_c) = \partial_c \int_{-\infty}^{\infty} \eta_c(x)\partial_x \varphi_c(x) dx \tag{3.26}$$

we deduce (3.22) by substituting (3.24) and (3.25) in (3.26). Finally, one also deduces from (3.24) and (3.25) that $\partial_x Q_c$ and $\partial_c Q_c$ are not co-linear. This completes the proof of Lemma 3.10. $\square$

Let us now come back to the proof of Proposition 3.6. We argue by contradiction. Suppose that there exists a sequence (U^n) such that

$$|U^n|_{X^0} = 1, \quad \lim_{n\to\infty} (L_c U^n, U^n) = 0, \quad (U^n, JR_c\partial_x Q_c) = (U_1^n, \partial_x \eta_c) = 0\,. \tag{3.27}$$

Then (up to the extraction of a subsequence), there exists $\tilde{U}_1 \in H^1$ such that (U_1^n) converges weakly in H^1 to $\tilde{U}_1$, and, by using Remark 3.5, there exists $\tilde{U}_2 \in \dot{H}_*^{\frac{1}{2}}$ such that U_2^n converges weakly in L^2_{loc} towards $\tilde{U}_2$ and $\mathfrak{P}U_2^n$ converges weakly in L^2 towards $\mathfrak{P}\tilde{U}_2$. Next, we set $\tilde{U} = (\tilde{U}_1, \tilde{U}_2) \in X^0$. We have that $\tilde{U}$ satisfies the same orthogonality conditions as U^n:

$$(\tilde{U}, JR_c\partial_x Q_c) = (\tilde{U}_1, \partial_x\eta_c) = 0\,. \tag{3.28}$$

Indeed, the second assertion is a direct consequence of the weak L^2 convergence of U_1^n to $\tilde{U}_1$. In order to prove the first one, we write

$$0 = (U^n, JR_c\partial_x Q_c) = (U_1^n, \partial_x\varphi_c - Z_c\partial_x\eta_c) - (U_2^n, \partial_x\eta_c)\,.$$

From the weak L^2 convergence of (U_1^n) to $\tilde{U}_1$, we obtain that

$$(U_1^n, \partial_x\varphi_c - Z_c\partial_x\eta_c) \to (\tilde{U}_1, \partial_x\varphi_c - Z_c\partial_x\eta_v)$$

and by observing that

$$(U_2^n, \partial_x\eta_c) = -\Big(\mathfrak{P}U_2^n, (1-\partial_x^2)^{1/4}\eta_c\Big), \tag{3.29}$$

since $(1-\partial_x^2)^{1/4}\eta_c \in L^2$, we also get by weak L^2 convergence that

$$(U_2^n, \partial_x\eta_c) \to -\Big(\mathfrak{P}\tilde{U}_2, (1-\partial_x^2)^{1/4}\eta_c\Big) = (\tilde{U}_2, \partial_x\eta_c)$$

(the last scalar product is well defined thanks to the exponential decay of $\partial_x\eta_c$). This proves that $\tilde{U}$ verifies (3.28).

Let us now complete the proof of Proposition 3.6. Let us set

$$\mathcal{E}_c = -b\partial_x\big(\zeta_c\partial_x\,\cdot\,\big) + g + (v_c - c)\partial_x Z_c, \quad \zeta_c = (1 + |\partial_x\eta_c|^2)^{-\frac{1}{2}},$$

we shall first conjugate to get a leading order part with constant coefficients:

$$\tilde{L}_c = AL_cA, \quad A = \begin{pmatrix} m_c & 0 \\ 0 & 1 \end{pmatrix}, \quad m_c = \frac{1}{\sqrt{\zeta_c}}$$

hence

$$\tilde{L}_c = \begin{pmatrix} m_c\mathcal{E}_c(m_c\cdot) & (v_c - c)\, m_c\partial_x \\ -\,\partial_x((v_c-c)m_c\cdot) & G[\eta_c] \end{pmatrix}.$$

Let us set

$$V_1^n = \frac{1}{m_c}U_1^n, \quad V_2^n = U_2^n,$$

so that

$$\big(L_cU^n, U^n\big) = \big(\tilde{L}_cV^n, V^n\big). \tag{3.30}$$

Note that we still have that V_1^n is bounded in H^1. We can thus assume (after extracting a subsequence) that V_1^n converges strongly in H^s_{loc} for every $s < 1$ towards $\tilde{V}_1 = 1/m_c\tilde{U}_1$. For convenience, we also set $\tilde{V}_2 = \tilde{U}_2$ so that V_2^n converges weakly in L^2_{loc} towards $\tilde{V}_2$ and that $\mathfrak{P}V_2^n$ converges weakly in L^2 towards $\mathfrak{P}\tilde{V}_2$.

Next, we observe that we can write the decomposition

$$\tilde{L}_c = L_1 + K_1,\ L_1 = \begin{pmatrix} -b\partial_x^2 + g & -c\partial_x \\ c\partial_x & G[\eta_c] \end{pmatrix} \tag{3.31}$$

and

$$K_1 = \begin{pmatrix} (v_c - c)m_c^2\partial_x Z_c + g(m_c - 1) + b\, m_c\partial_x(m_c\zeta_c)\partial_x + b\, m_c\partial_x\big(\zeta_c\partial_x m_c\,\cdot\big) & \big(v_c m_c - c(m_c-1)\big)\partial_x \\ -\partial_x((v_c m_c + c(1-m_c))\cdot) & 0 \end{pmatrix}.$$

For K_1, which is a relatively compact perturbation, we obtain

$$\lim_{n\to\infty}(K_1 V^n, V^n) = (K_1\tilde{V}, \tilde{V}). \tag{3.32}$$

Indeed, we can use the same trick as in (3.29), the exponential decay of v_c, $\partial_x Z_c$, $m_c - 1$, $\partial_x m_c$, $\partial_x^2 m_c$ and the fact that V_1^n converges strongly in H^s_{loc} for every $s < 1$.

To analyze L_1, we use the following factorization of its associated quadratic form: for every V

$$(L_1 V, V) = \big((-b\,\partial_x^2 + g - c^2 M^{-1})V_1, V_1\big) + \big(M\big(\partial_x V_2 - cM^{-1}V_1\big), \partial_x V_2 - cM^{-1}V_1\big)$$

where

$$M = -\partial_x^{-1}G[\eta_c]\partial_x^{-1}.$$

Note that M is a well-defined operator (of order -1), positive and invertible thanks to Lemma 4.5 of [24]. Let us set

$$M_0 = -\partial_x^{-1}G[0]\partial_x^{-1},$$

we can rewrite

$$(L_1 V, V) = (\tilde{L}_1 V, V) - c^2\big((M^{-1} - M_0^{-1})V_1, V_1\big).$$

with

$$(\tilde{L}_1 V, V) = \big((-b\,\partial_x^2 + g - c^2 M_0^{-1})V_1, V_1\big) + \big(M\big(\partial_x V_2 - cM^{-1}V_1\big), \partial_x V_2 - cM^{-1}V_1\big). \tag{3.33}$$

Note that $M - M_0$ is a compact operator on H^1 (see the remark at the bottom of p. 295 and the top of p. 296 in [24]), consequently, we have

$$\lim_n \big((M - M_0)V_1^n, V_1^n\big) = \big((M - M_0)\tilde{V}_1, \tilde{V}_1\big). \tag{3.34}$$

To pass to the limit in $(\tilde{L}_1 V^n, V^n)$, we note that $\tilde{L}_1$ is a non-negative operator. Indeed, M is nonnegative and the first part of $\tilde{L}_1$ is a Fourier multiplier with symbol

$$m(\xi) = b\xi^2 + g - c^2\frac{\xi}{\tanh H\xi} = \frac{c^2}{H}\Big(\beta(H\xi)^2 + \alpha - \frac{H\xi}{\tanh H\xi}\Big)$$

which is positive since $\beta > 1/3$ and $\alpha > 1$ thanks to (1.7). Consequently, we get from the weak convergence properties that

$$\liminf_{n\to\infty}(\tilde{L}_1 V^n, V^n) \geq (\tilde{L}_1\tilde{V}, \tilde{V}).$$

Gathering the previous transformations, we thus get that

$$\liminf_{n\to\infty}\big(L_c U^n, U^n\big) = \liminf_{n\to\infty}\big(\tilde{L}_c V^n, V^n\big) \geq \big(\tilde{L}_1\tilde{V}, \tilde{V}\big) - c^2\big((M - M_0)\tilde{V}_1, \tilde{V}_1\big) + \big(K_1\tilde{V}, \tilde{V}\big) \tag{3.35}$$

$$= \big(\tilde{L}_c\tilde{V}, \tilde{V}\big) = \big(L_c\tilde{U}, \tilde{U}\big). \tag{3.36}$$

Consequently, thanks to (3.27), we find that $\big(L_c\tilde{U}, \tilde{U}\big) \leq 0$ Since $\tilde{U}$ verifies (3.28), we get thanks to Lemma 3.8 that $\tilde{U} \in 0_{X^0}$ that is to say $\tilde{U} = (0, \lambda)^t$ for some constant $\lambda \in \mathbb{R}$. Since K_1 is such that $(K_1((0,\lambda)^t), (0,\lambda)^t) = 0$, we obtain by using again (3.30), (3.31), (3.32), (3.34) that as $n \to \infty$,

$$(L_c U^n, U^n) = (\tilde{L}_1 V^n, V^n) + o(1).$$

Moreover, since the symbol $m(\xi)$ is positive and the operator M is non-negative we get from (3.33) that for some $c_0 > 0$ independent of n, we have

$$(L_c U^n, U^n) \geq c_0\|V_1^n\|^2_{H^1} + o(1).$$

This yields that V_1^n converges strongly to 0 in H^1. This is sufficient to also obtain that

$$\lim_n (\partial_x V_2^n, V_1^n) = 0$$

and hence we find by using again (3.30), (3.31) that

$$\big(L_c U^n, U^n\big) = (G[\eta_c]U_2^n, U_2^n) + o(1).$$

Thanks to (3.4) in Proposition 3.1, this implies that $\mathfrak{P}U_2^n$ converges strongly to zero in L^2. We have thus obtained that $\lim_n \|U^n\|_{X^0} = 0$ which contradicts the assumption that $\|U^n\|_{X^0} = 1$ in (3.27).

This ends the proof of Proposition 3.6. $\square$

From Proposition 3.6, one can get that

Proposition 3.11. *For every $U \in X^0$ there exists a unique decomposition*

$$U = \alpha J R_c \partial_x Q_c + \beta R_c \partial_x Q_c + V \tag{3.37}$$

with $V \in X^0$ such that

$$(V, JR_c\partial_x Q_c) = (V_1, \partial_x \eta_c) = 0. \tag{3.38}$$

Moreover, there exists $c_0 > 0$ and $C > 0$ such that for every $U \in X^0$ written under the form (3.37), one has

$$(L_c U, U) \geq c_0 |V|_{X^0}^2 - C|\alpha|^2.$$

Note that in the decomposition (3.37), V is not orthogonal to the $R_c\partial_x Q_c$. This decomposition has better properties than the orthogonal decomposition that one would get from proposition 3.6 by choosing V orthogonal to $JR_c\partial_x Q_c$ and $(\partial_x \eta_c, 0)^t$. The reason is that $R_c \partial_x Q_c$ is in the kernel of L_c while $(\partial_x\eta_c, 0)^t$ is not.

Proof of Proposition 3.11. If

$$U = \alpha J R_c \partial_x Q_c + \beta R_c \partial_x Q_c + V \tag{3.39}$$

with V satisfying (3.38) then α and β are necessarily determined by

$$\alpha = \frac{(U, JR_c\partial_x Q_c)}{|JR_c \partial_x Q_c|_{L^2}^2}, \quad \beta = \frac{(U, (\partial_x\eta_c, 0)^t)}{(R_c\partial_x Q_c, (\partial_x \eta_c, 0)^t)} - \alpha (JR_c\partial_x Q_c, (\partial_x\eta_c, 0)^t) \tag{3.40}$$

and thus are well-defined since $(R_c\partial_x Q_c, (\partial_x\eta_c, 0)^t) = |\partial_x \eta_c|_{L^2}^2 \neq 0$. This proves the uniqueness and one directly verifies that with α, β defined by (3.40), the function V in (3.39) satisfies the orthogonality conditions (3.38).

Next, since $L_c R_c \partial_x Q_c = \Lambda_c \partial_x Q_c = 0$, we get by using the decomposition (3.39) that

$$(L_c U, U) = \alpha^2 (L_c J R_c \partial_x Q_c, JR_c\partial_x Q_c) + 2\alpha (V, L_c J R_c \partial_x Q_c) + (L_c V, V).$$

Therefore, by using Proposition 3.6 for V, we obtain

$$(L_c U, U) \geq \frac{c_0}{2} \|V\|_{X^0}^2 - C\alpha^2.$$

This completes the proof of Proposition 3.11. $\square$

As a simple corollary, we can get stability estimates in X^0 for the linearized equation

$$\partial_t U = J L_c U, \quad U_{/t=0} = U^0. \tag{3.41}$$

Corollary 3.12. *There exists $C > 0$ such that for every $U^0 \in X^0$, the solution of $U(t)$ of (3.41) satisfies the estimate*

$$|U(t)|_{X^0} \leq C(1 + |t|)|U^0|_{X^0}.$$

Proof. From the explicit expression (3.40) for α, we get $\partial_t\alpha = 0$, and therefore

$$|\alpha(t)| = |\alpha(0)| \leq C(\|U_1^0\|_{L^2} + |(U_2^0, \partial_x\eta_c)|) \leq C|U^0|_{X^0}\,. \tag{3.42}$$

Since U solves (3.41), we observe that $\partial_t(L_cU, U) = 0$. Therefore, we get from Proposition 3.11 and (3.42) that

$$|V(t)|_{X^0}^2 \leq C\big((L_cU^0, U^0) + |\alpha(t)|^2\big) \leq C|U^0|_{X^0}^2\,.$$

Moreover, from the explicit expression (3.40) for β, we find that

$$\partial_t\beta = \frac{(JL_cU, (\partial_x\eta_c, 0)^t)}{(R_c\partial_xQ_c, (\partial_x\eta_c, 0)^t)} = \frac{(JL_c(\alpha JR_c\partial_xQ_c + V), (\partial_x\eta_c, 0)^t)}{(R_c\partial_xQ_c, (\partial_x\eta_c, 0)^t)}.$$

Hence

$$|\beta(t)| \leq C|U^0|_{X^0} \;+ C\int_0^t (|\alpha(s)| + |V(s)|_{X^0})ds \leq C(1+|t|)|U^0|_{X^0}.$$

This completes the proof of Corollary 3.12 $\square$

4. Error produced by a sum of solitary waves

We shall now begin the construction that will allow to get Theorem 1.2. We denote the two different soliton solutions for the water-wave system as (1.5) as

$$\begin{aligned} Q_{c_1}(x-c_1t) = Q_1(x-c_1t) &= (\eta_1(x-c_1t), \varphi_1(x-c_1t))^t \\ Q_{c_2}(x-h-c_2t) = Q_2(x-h-c_2t) &= (\eta_2(x-h-c_2t), \varphi_2(x-h-c_2t))^t, \end{aligned}$$

and we set

$$M(t,x) = Q_1(x-c_1t) + Q_2(x-h-c_2t) := (\eta_M(t,x), \varphi_M(t,x))^t$$

to be the superposition of the two solitary waves. Sometimes we shall simply write $Q_1 = (\eta_1, \varphi_1)^t$ and $Q_2 = (\eta_2, \varphi_2)^t$ for convenience. Thanks to Theorem 1.1, we have

$$\exists d > 0,\ \forall\alpha,\ \exists C_\alpha,\ \forall x \in \mathbb{R}, \quad |\partial_{t,x}^\alpha\eta_i| \leq C_\alpha e^{-d|x-c_it-(i-1)h|}, \quad i = 1,\, 2 \tag{4.1}$$

and

$$\exists d > 0,\ \forall\alpha \in \mathbb{N},\ \alpha \geq 1,\ \exists C_\alpha,\ \forall x \in \mathbb{R}, \quad |\partial_{t,x}^\alpha\varphi_i| \leq C_\alpha e^{-d|x-c_it-(i-1)h|}, \quad i = 1,\, 2. \tag{4.2}$$

Note that φ_i is bounded but not exponentially decaying, nervetheless, derivatives of φ_i are exponentially decaying. In this section, we shall establish that M solves the water-waves system (1.5) up to a small exponentially decaying term:

Proposition 4.1. *The two-soliton $M(t,x)$ solves:*

$$\begin{cases} \partial_t\eta_M = G[\eta_M]\varphi_M + R_1 \\ \partial_t\varphi_M = -\frac{1}{2}|\partial_x\varphi_M|^2 + \frac{1}{2}\frac{(G[\eta_M]\varphi_M + \partial_x\varphi_M\partial_x\eta_M)^2}{1+|\partial_x\eta_M|^2} - g\eta_M + b\partial_x\left(\frac{\partial_x\eta_M}{\sqrt{1+|\partial_x\eta_M|^2}}\right) + R_2 \end{cases}$$

where the remainder $R_M(t,x) := (R_1, R_2)^t$ has an exponential decay in time, that is, there exist constants C_s and $\epsilon_0 > 0$ such that for any $s \geq 0$

$$|R_M(t)|_{E^s} \leq C_s e^{-\epsilon_0 h}e^{-\epsilon_0(c_2-c_1)t}, \quad \forall t \geq 0.$$

Let us recall the notation

$$|U(t)|_{E^s} = \sum_{|\alpha|\leq s} |\partial_{t,x}^\alpha U|_{L^2}.$$

Note that in the sequel we use again both ε and ϵ for different parameters.

In order to prove Proposition 4.1, the main difficulty is to study the interaction of the two solitary waves via the Dirichlet-Neumann operator, i.e we need to study for example $G[\eta_1]\varphi_2$, thus the crucial ingredient will be Proposition 3.2.

4.1. **Proof of Proposition 4.1.** The basic idea is that the interaction between the solitary waves is weak because they are localized and far way with different speeds. We shall use many times in the proof the following elementary lemma:

Lemma 4.2. *For $c_2 > c_1$, $\epsilon > 0$ and $\epsilon_0 \in (0, \epsilon)$, there exists $C > 0$ such that for every $h \geq 0$, $t \geq 0$,*

$$\int_{\mathbb{R}} e^{-\epsilon|x-c_1t|}e^{-\epsilon|x-h-c_2t|}\,dx \leq Ce^{-\epsilon h}e^{-\epsilon_0(c_2-c_1)t}.$$

Proof. It suffices to decompose the integration domain in the three regions $x \geq h + c_2t$, $x \leq c_1t$ and $c_1t \leq x \leq h + c_2t$. □

Let us prove Proposition 4.1. Since Q_1 and Q_2 are two solutions of (1.5), we can sum up the first equations of the two systems to get

$$\begin{aligned}\partial_t\eta_M &= \partial_t\eta_1 + \partial_t\eta_2 = G[\eta_1]\varphi_1 + G[\eta_2]\varphi_2\\ &= G[\eta_M]\varphi_M + G[\eta_1]\varphi_1 - G[\eta_M]\varphi_1 + G[\eta_2]\varphi_2 - G[\eta_M]\varphi_2.\end{aligned}$$

So we have the first equation for M:

$$\partial_t\eta_M = G[\eta_M]\varphi_M + R_1$$

with $R_1 = G[\eta_1]\varphi_1 - G[\eta_M]\varphi_1 + G[\eta_2]\varphi_2 - G[\eta_M]\varphi_2$. Next we need to estimate R_1. Using the shape-derivative formula for the Dirichlet-Neumann operator (see (5) Proposition 3.1), we can compute that

$$\begin{aligned}G[\eta_1]\varphi_1 - G[\eta_M]\varphi_1 &= -\int_0^1 D_\eta G[\eta_1 + s\eta_2]\varphi_1 \cdot \eta_2 ds\\ &= \int_0^1 [G[\eta_1 + s\eta_2](\eta_2 Z_{1s}) + \partial_x(\eta_2(\partial_x\varphi_1 - Z_{1s}\partial_x(\eta_1 + s\eta_2))]\,ds\end{aligned}$$

where

$$Z_{1s} = \frac{G[\eta_s]\varphi_1 + \partial_x\eta_s\partial_x\varphi_1}{1 + |\partial_x\eta_s|^2}$$

with the notation $\eta_s = \eta_1 + s\eta_2$. Next, we can use Corollary 3.4 to get

$$|\partial_{t,x}^\alpha(G[\eta_s]\varphi_1)| \leq C_\alpha e^{-\epsilon|x-c_1t|}. \quad \text{for} \quad \alpha \in \mathbb{N},$$

Indeed, since the solitary waves depend on $x - c_it$, one can always convert a time derivative into a space derivative. We thus get

$$|\partial_{t,x}^\alpha Z_{1s}| \leq C_\alpha e^{-\epsilon|x-c_1t|}, \quad \text{for} \quad \alpha \in \mathbb{N}.$$

With these estimates and using Lemma 4.2, we get

$$\begin{aligned}|G[\eta_1]\varphi_1 - G[\eta_M]\varphi_1|_{E^s} &\leq C\int_0^1 (|\eta_2 Z_{1s}|_{H^{s+1}} + |\eta_2\partial_x\varphi_1|_{E^{s+1}} + |\eta_2 Z_{1s}|_{E^{s+1}})\,ds\\ &\leq C_s e^{-\epsilon_0 h}e^{-\epsilon_0(c_2-c_1)t}, \quad \text{for} \quad t \geq 0.\end{aligned}$$

Similarly we have the estimate for $|G[\eta_2]\varphi_2 - G[\eta_M]\varphi_2|_{E^s}$. Summing these two estimates up leads to

$$|R_1|_{E^s} \leq C_s e^{-\epsilon_0 h}e^{-\epsilon_0(c_2-c_1)t}, \quad \text{for} \quad t \geq 0.$$

Now we deal with the equation for φ_M. From the second equation of system (1.5) for both Q_1 and Q_2 one can write down that

$$\begin{aligned}\partial_t\varphi_M &= \partial_t\varphi_1 + \partial_t\varphi_2\\ &= -\frac{1}{2}|\partial_x\varphi_M|^2 + \frac{1}{2}\frac{(G[\eta_M]\varphi_M + \partial_x\varphi_M\partial_x\eta_M)^2}{1 + |\partial_x\eta_M|^2} - g\eta_M + b\partial_x\left(\frac{\partial_x\eta_M}{\sqrt{1 + |\partial_x\eta_M|^2}}\right) + R_2\end{aligned}$$

where $R_2 = R_{21} + R_{22} + R_{23}$ with

$$
\begin{aligned}
R_{21} &= \frac{1}{2}|\partial_x \varphi_M|^2 - \sum_{i=1}^{2} \frac{1}{2}|\partial_x \varphi_i|^2 = \partial_x \varphi_1 \partial_x \varphi_2, \\
R_{22} &= -\frac{1}{2} \frac{(G[\eta_M]\varphi_M + \partial_x \varphi_M \partial_x \eta_M)^2}{1 + |\partial_x \eta_M|^2} + \sum_{i=1}^{2} \frac{1}{2} \frac{(G[\eta_i]\varphi_i + \partial_x \varphi_i \partial_x \eta_i)^2}{1 + |\partial_x \eta_i|^2}, \\
R_{23} &= -b\partial_x \left(\frac{\partial_x \eta_M}{\sqrt{1 + |\partial_x \eta_M|^2}} \right) + b \sum_{i=1}^{2} \partial_x \left(\frac{\partial_x \eta_i}{\sqrt{1 + |\partial_x \eta_i|^2}} \right).
\end{aligned}
$$

With the same arguments that we have used for the estimate of R_1, we get that R_2 also satisfies the same exponential-decay estimate

$$
|R_2|_{E^s} \le C_s e^{-\epsilon_0 h} e^{-\epsilon_0 (c_2 - c_1) t}, \quad \text{for} \quad t \ge 0.
$$

This ends the proof of Proposition 4.1. □

5. Construction of an approximate solution

If we take $\delta = e^{-\epsilon_0 h} > 0$, δ will be small enough if we choose $h > 0$ large enough later. The remainder R_M in the system for $M(t,x)$ can be rewritten as $R_M = \delta \tilde{R}_M$ with

$$
|\tilde{R}_M|_{H^s} \le C_s e^{-\epsilon_0 (c_2 - c_1) t}, \quad \text{for} \quad t \ge 0. \tag{5.1}
$$

Let

$$
V(t,x) = \sum_{l=1}^{N} \delta^l V_l(t,x)
$$

with unknowns $V_l(t,x)$ to be constructed later. We want to show that $U^a(t,x) = M(t,x) + V(t,x)$ is an approximate solution for the water-wave system under the following sense:

Proposition 5.1. *For any $N \in \mathbb{N}$, there exists*

$$
U^a(t,x) = M(t,x) + V(t,x) = M(t,x) + \sum_{l=1}^{N} \delta^l V_l, \quad \textit{with} \quad V_l \in C^\infty(\mathbb{R}_+, H^\infty(\mathbb{R}))
$$

and a small constant $\delta = e^{-\epsilon_0 h} > 0$ such that for every l, one has the estimates

$$
|V_l(t)|_{E^k} \le h^{\frac{2l-1}{4}} C_{k,l} e^{-l\epsilon_0 (c_2 - c_1) t}, \qquad \forall t \ge 0
$$

with some constant $C_{k,l}$. Moreover, U^a is an approximate solution of (1.6) in the sense that

$$
\partial_t U^a - \mathcal{F}(U^a) = R_{ap}
$$

where R_{ap} satisfies the estimate

$$
|R_{ap}|_{E^s} \le C_{N,s} h^{\frac{2N+1}{4}} \delta^{N+1} e^{-\epsilon_0 (N+1)(c_2 - c_1) t}, \quad \textit{for} \quad t \ge 0,
$$

In order to prove this proposition, we use again the Taylor expansion of $\mathcal{F}$

$$
\mathcal{F}(M + V) = \mathcal{F}(M) + \sum_{l=1}^{N} \frac{1}{l!} D^l \mathcal{F}[M](V, \dots, V) + R_{N,\delta}(V)
$$

where the first derivative of $\mathcal{F}$ is $D\mathcal{F} = J\Lambda[M]$ where

$$
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda[M] = \begin{pmatrix} -\mathcal{P}_M + g + Z_M G[\eta_M](Z_M \cdot) + Z_M \partial_x v_M & v_M \partial_x - Z_M G[\eta_M] \\ -\partial_x (v_M \cdot) - G[\eta_M](Z_M \cdot) & G[\eta_M] \end{pmatrix}
$$

with the notations

$$Z_M = Z[\eta_M, \varphi_M],\ v_M = v[\eta_M, \varphi_M],\ \mathcal{P}_M = \mathcal{P}[\eta_M, \varphi_M].$$

The notations Z and v are introduced in Proposition 3.1 while $\mathcal{P}$ is defined by

$$\mathcal{P}[\eta, \varphi]u = b\partial_x\Big((1 + (\partial_x\eta)^2)^{-\frac{3}{2}}\partial_x u\Big).$$

We shall also introduce the notations

$$Z_i = Z[Q_i], \quad v_i = v[Q_i], \quad \mathcal{P}_i = \mathcal{P}[Q_i], \quad i = 1, 2.$$

Plugging $V(t,x) = \sum_{l=1}^N \delta^l V_l(t,x)$ into the system leads to linear problems with source terms for V_k. The system for V_1 is

$$\partial_t V_1 - J\Lambda[M]V_1 = -\tilde{R}_M, \tag{5.2}$$

the system for V_2 is

$$\partial_t V_2 - J\Lambda[M]V_2 = \frac{1}{2}D^2\mathcal{F}[M](V_1, V_1)$$

and the general equation for V_l is

$$\partial_t V_l - J\Lambda[M]V_l = \sum_{p=2}^{l} \sum_{\substack{1\le l_1,\dots,l_p\le l-1\\ l_1+\cdots+l_p=l}} \frac{1}{p!}D^p\mathcal{F}[M](V_{l_1}, \dots, V_{l_p}). \tag{5.3}$$

Before solving these systems, let us fix and recall some notations to be used in the remaining part of the paper. For $U(t,x) = (U_1, U_2)^t$, we define

$$\begin{aligned} |U(t)|_{X^k}^2 &= \sum_{0\le\alpha+\beta\le k} \Big(|\partial_t^\alpha\partial_x^\beta U_1(t,\cdot)|_{H^1(\mathbb{R})}^2 + |\partial_t^\alpha\partial_x^\beta U_2(t,\cdot)|_{\dot{H}_*^{\frac{1}{2}}(\mathbb{R})}^2\Big)\\ |U(t)|_{W^k} &= \sup_{\alpha+\beta\le k} |\partial_t^\alpha\partial_x^\beta U(t,\cdot)|_{L^\infty}\,, \end{aligned}$$

where

$$|\varphi|_{\dot{H}_*^{\frac{1}{2}}(\mathbb{R})}^2 = |\mathfrak{P}\varphi|_{L^2(\mathbb{R})}^2\,.$$

Note that X^0 is the natural energy space for the water-waves system.

5.1. **The homogeneous linear system.** The main ingredient in the proof of Proposition 5.1 will be a rather precise estimate of the growth rate of the fundamental solution of the linear homogeneous equation

$$\partial_t V - J\Lambda[M]V = 0 \tag{5.4}$$

which corresponds to the linearization of the water-waves system about the multi-solitary wave M. As before, with

$$R = \begin{pmatrix} 1 & 0\\ -Z_M & 1\end{pmatrix}$$

we can perform the change of unknowns $U = RV$ to get a simpler linear system which is equivalent to (5.4)

$$\partial_t U - JL[M]U = 0 \tag{5.5}$$

where

$$L[M] = \begin{pmatrix} -\mathcal{P}_M + g + a_M & v_M\partial_x\\ -\partial_x(v_M\cdot) & G[\eta_M]\end{pmatrix}$$

is a self-adjoint operator with the notation $a_M = a[M] = v_M\partial_x Z_M + \partial_t Z_M$ and we will take $L_M = L[M], G_M = G[\eta_M]$ for convenience. We shall also use the notations

$$a_i = a[Q_i], \quad L_i = L[Q_i], \quad i = 1, 2$$

and so on. Note that this is a short hand for:

$$L_1u(t,x) = L[Q_{c_1}(x - c_1t)]u(t,x), \quad L_2u(t,x) = L[Q_{c_2}(x - h - c_2t)]u(t,x).$$

The main result of this section will be:

Theorem 5.2. *For $\varepsilon \in (0, \varepsilon^*]$, there exists h_0 and $C > 0$ such that for every $h \geq h_0$, the solution of (5.5) with initial datum $U(\tau)$ at $t = \tau$ satisfies the estimate*

$$|U(t)|_{X^k} + \sum_{\alpha \leq k} |\partial_t^\alpha U_2|_{L^2} \leq h^{\frac{1}{4}} C_k(|U(\tau)|_{X^k} + \sum_{\alpha \leq k} |\partial_t^\alpha U_2(\tau)|_{L^2}) \big(1 + \epsilon_0(c_2 - c_1)(t - \tau)\big)^k e^{\epsilon_0(c_2 - c_1)(t-\tau)/2},$$
$$\forall t \geq \tau \geq 0.$$

Remark 5.3. *Note that in the above estimate, the right hand-side can be expressed in terms of usual Sobolev regularity of the initial data by using the system (5.5) to express the time derivatives of the solution at the initial time $t = \tau$. Let us denote by $S_M(t, \tau)$ the fundamental solution of the (non-autonomous) system (5.5). The estimate of Theorem 5.2 can thus be rewritten under the form: for every $k \leq 0$, there exists $C_k(h, \epsilon_0) > 0$ such that*

$$|S_M(t, \tau)U|_{E^k} \leq C_k h^{\frac{1}{4}} (1 + \epsilon_0(t - \tau)^k)|U|_{H^{s(k)}} e^{\epsilon_0(c_2 - c_1)(t-\tau)/2}, \forall t \geq \tau \geq 0. \tag{5.6}$$

We do not need for our argument a sharp estimate of the number $s(k)$. A straightforward possibility, is to take $s(k) = 2k + 1$ since a time derivative of the solution always costs at most two space derivatives of the initial data. The meaning of this result is that when each solitary wave is stable (this corresponds to ε small), then by choosing h sufficiently large, we can get an arbitrary slow exponential growth rate for the fundamental solution of (5.5), the special form of this growth rate that we have chosen is just one that it is sufficient for the proof of Proposition 5.1. Note that the shape that we have chosen is linked in particular to the decay rate of the remainder R_M in Proposition 4.1

We shall split the proof of the above estimate into many steps. For notational convenience, we shall give the proof only in the case $\tau = 0$ which gives the worse constraint of h_0. The general case can be deduced from this one by replacing t by $t - \tau$, x by $x - c_1\tau$ and thus in the multi-solitary wave M, h by $\tilde{h} = h + (c_2 - c_1)\tau \geq h$.

During the proof C is a positive number which change from line to line but which is independent of h for $h \geq 1$ and t for $t \geq 0$.

We shall first define a decomposition of unity in order to localize our energy estimates in the vicinity of each solitary wave. We take $\chi^0 \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\chi^0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x \geq 1 \end{cases}$$

and we define

$$\tilde{\chi}_1(t,x) = \chi^0\Big(\frac{x - \frac{h}{4} - c_m t}{\frac{h}{4}}\Big), \; c_m = \frac{c_1 + c_2}{2}, \quad \tilde{\chi}_2(t,x) = 1 - \tilde{\chi}_1(t,x).$$

Finally, we take

$$\chi_1(t,x) = \frac{\tilde{\chi}_1}{(\tilde{\chi}_1^2 + \tilde{\chi}_2^2)^{\frac{1}{2}}}, \quad \chi_2(t,x) = \frac{\tilde{\chi}_2}{(\tilde{\chi}_1^2 + \tilde{\chi}_2^2)^{\frac{1}{2}}}. \tag{5.7}$$

Note that these functions are smooth and bounded and defined such that $\chi_1^2 + \chi_2^2 = 1$.

The main properties of these functions that we shall use are:

Lemma 5.4. *The above χ_i satisfy*

$$\forall \beta,\ |\beta| \geq 1,\ |\partial_{t,x}^{\beta} \chi_i(t,x)| \lesssim \frac{1}{h^{|\beta|}}, \quad i = 1,\, 2. \tag{5.8}$$

Moreover, for any $\epsilon > 0$, we have

$$|e^{-\epsilon|x-c_1t|}\chi_2| \leq \frac{C_\epsilon}{h}, \quad |e^{-\epsilon|x-h-c_2t|}\chi_1| \leq \frac{C_\epsilon}{h}, \quad \forall t \geq 0,\, x \in \mathbb{R}, h \geq 1 \tag{5.9}$$

for some $C_\epsilon > 0$.

Proof. The estimate (5.8) is clear. For the first one in (5.9), we observe that χ_2 is supported in $x \geq c_m t + h/4$. Since in this region $x - c_1 t \geq (c_m - c_1)t + h/4 \geq 0$, we immediately get

$$|e^{-\epsilon|x-c_1t|}\chi_2| \lesssim e^{-\epsilon(c_m-c_1)t} e^{-\epsilon\frac{h}{4}} \lesssim \frac{1}{h}.$$

The second estimate follows by observing that χ_1 is supported in $x \leq h/2 + c_m t$. This completes the proof of Lemma 5.4. □

5.2. **Lower-order energy estimate.** The first step in the proof of Theorem 5.2 will be to prove the estimate for $k = 0$. We shall thus prove

Proposition 5.5. *Under the assumptions of Theorem 5.2, we have the estimate:*

$$|U(t)|_{X^0}^2 + |U_2(t)|_{L^2}^2 \leq C\Big(h^{\frac{1}{2}}|U(\tau)|_{X^0}^2 + |U_2(\tau)|_{L^2}^2\Big)e^{\epsilon_0(c_2-c_1)(t-\tau)/4}, \qquad \forall t \geq \tau \geq 0.$$

Proof. Again, we shall give the proof only for $\tau = 0$.

Let us consider the energy functional

$$E_1(U(t)) = (L_M U,\, U) - c_1(A\chi_1 U,\, \chi_1 U) - c_2(A\chi_2 U,\, \chi_2 U)$$

with A the symmetric operator

$$A = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} = \partial_x J.$$

We shall prove the E_1 is an almost conserved quantity with some positivity property thanks to Proposition 3.11 applied to each solitary wave.

We first write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E_1(U(t)) &= (\partial_t U,\, L_M U) + \frac{1}{2}([\partial_t,\, L_M]U,\, U) - c_1(A\chi_1\partial_t U,\, \chi_1 U) \\
&\quad - c_2(A\chi_2\partial_t U, \chi_2 U) - c_1(A(\partial_t\chi_1)U,\, \chi_1 U) - c_2(A(\partial_t\chi_2)U,\, \chi_2 U) \\
&:= I_1 + I_2 + \cdots + I_6.
\end{aligned} \tag{5.10}$$

We will estimate these terms one by one. Towards this, let us set

$$U^1 = \chi_1 U, \quad \text{and} \quad U^2 = \chi_2 U$$

with $\sum_{i=1,2}\chi_i^2 = 1$ from (5.7). We shall use very often the following norm equivalence properties.

Lemma 5.6. *There exists $C > 0$ such that for every $h \geq 1$, we have*

$$|U|_{X^0}^2 \leq C(\sum_{i=1,2}|U^i|_{X^0}^2 + \frac{1}{h}|\kappa(D)U_2|_{L^2}^2)$$

and also

$$\sum_{i=1,2}|U^i|_{X^0}^2 \leq C(|U|_{X^0}^2 + \frac{1}{h}|\kappa(D)U_2|_{L^2}^2)$$

with $\hat{\kappa}(\xi)$ is a smooth cut-off function with $\hat{\kappa}(\xi) = 1$ around $\xi = 0$.

Proof. Let us prove the first estimate. From Lemma 5.4, we first easily get that

$$|U_1|_{H^1}^2 \leq C(\sum_{i=1,2} |\chi_i U_1|_{X^0}^2 + \frac{1}{h}|U_1|_{L^2}^2),$$

For the estimate on U_2, we use the basic commutator estimate

$$|[\mathfrak{P},\, f]g|_{L^2} \lesssim |\partial_x f|_{L^\infty}|g|_{L^2}. \tag{5.11}$$

Indeed, one can write $[\mathfrak{P},\, f]g = (1-\partial_x^2)^{-\frac{1}{4}}(\partial_x f g) + [(1-\partial_x^2)^{-\frac{1}{4}}, f]\partial_x g$. The L^2 norm of the first term is clearly bounded by the right hand-side of (5.11) while the L^2 norm of the second can be bounded by $C|\partial_x f|_{L^\infty}|g|_{L^2}$ by invoking [27, Proposition 3.6.B, estimate (3.6.35)]. This proves (5.11). Estimate (5.11) yields

$$\begin{aligned} |\mathfrak{P}U_2|_{L^2}^2 &\leq \sum_{i=1,2} |\mathfrak{P}\chi_i^2 U_2|_{L^2}^2 \leq \sum_{i=1,2} \big(|[\mathfrak{P},\, \chi_i]\chi_i U_2|_{L^2}^2 + |\chi_i \mathfrak{P}(\chi_i U_2)|_{L^2}^2\big) \\ &\leq C\sum_{i=1,2} (\frac{1}{h}|\chi_i U_2|_{L^2}^2 + |\mathfrak{P}(\chi_i U_2)|_{L^2}^2). \end{aligned}$$

The proof of the other estimate is similar. This ends the proof of Lemma 5.6 . □

Now we can go back to the study of (5.10). First of all, from the linear system (5.5), we have

$$I_1 = (JL_M U,\, L_M U) = 0.$$

Next, from the definition of L_M one can compute that

$$[\partial_t,\, L_M] = \begin{pmatrix} -[\partial_t,\, \mathcal{P}_M] + \partial_t a_M & (\partial_t v_M)\partial_x \\ -\partial_x((\partial_t v_M)\cdot) & [\partial_t,\, G_M] \end{pmatrix}$$

and so by combining this with the decomposition of unity leads to

$$2I_2 = ([\partial_t,\, L_M]\sum_{i=1,2} \chi_i^2 U,\, U) = \sum_{i=1,2} ([\partial_t,\, L_i]\chi_i U,\, \chi_i U) + 2I_{2R}$$

where

$$\begin{aligned} 2I_{2R} &= \sum_{i=1,2} \{([\partial_t,\, L_M - L_i]\chi_i U, \chi_i U) + ([[\partial_t,\, L_M],\, \chi_i]\chi_i U,\, U)\} \\ &= \sum_{i=1,2} \{((-[\partial_t,\, \mathcal{P}_M - \mathcal{P}_i] + \partial_t(a_M - a_i))\chi_i U_1,\, \chi_i U_1) + ((\partial_t(v_M - v_i))\partial_x(\chi_i U_2),\, \chi_i U_1) \\ &\quad +([\partial_t, G_M - G_i]\chi_i U_2,\, \chi_i U_2) - ([[\partial_t,\, \mathcal{P}_M],\, \chi_i]\chi_i U_1,\, U_1) + ([[\partial_t,\, G_M], \chi_i]\chi_i U_2,\, U_2)\}. \end{aligned}$$

For the remainder term I_{2R}, we have by using Lemma 5.4 the estimate

$$|I_{2R}| \leq \frac{1}{h}C(|U(t)|_{X^0}^2 + |U_2(t)|_{L^2}^2).$$

Indeed, let us study for example the first term in more details. We have

$$\mathcal{P}_M = b\partial_x\left(\frac{\partial_x\cdot}{(1+|\partial_x \eta_M|^2)^{\frac{3}{2}}}\right),$$

and thus

$$[\partial_t,\, \mathcal{P}_M - \mathcal{P}_i] = b\partial_x\left(\int_0^1 [\partial_t,\, \frac{\partial_x \eta_{is}\partial_x(\eta_M - \eta_i)}{(1+|\partial_x \eta_{is}|^2)^{\frac{3}{2}}} ds]\partial_x\cdot\right)$$

with the notation $\partial_x \eta_{is} = \partial_x \eta_i + s\partial_x(\eta_M - \eta_i)$. Consequently, by using again Lemma 5.4, we get

$$|[\partial_t,\, \mathcal{P}_M - \mathcal{P}_i]\chi_i U_1,\, \chi_i U_1)| \leq \frac{1}{h}C|U_1|_{H^1}^2.$$

One can estimate the other terms in the definition of I_{2R} by using the same arguments, in particular, for the terms involving the Dirichlet-Neumann operator, we also have

Lemma 5.7. *There exists a constant C such that the following commutator estimates hold*

$$([\partial_t,\, G_M - G_i]\chi_i U_2,\, \chi_i U_2) \le \frac{1}{h} C(|\mathfrak{P}U_2|_{L^2}^2 + |U_2|_{L^2}^2) \tag{5.12}$$

$$([[\partial_t,\, G_M],\, \chi_i]\chi_i U_2,\, U_2) \le \frac{1}{h} C(|\mathfrak{P}U_2|_{L^2}^2 + |U_2|_{L^2}^2) \tag{5.13}$$

where $\chi_i\,(i = 1, 2)$ are defined in (5.7).

This lemma will be proven in the appendix.

For the I_3 term in (5.10), we get by using the system (5.5) that

$$\begin{aligned} I_3 &= -c_1(A\chi_1 J L_M U,\, \chi_1 U) = c_1(\partial_x(\chi_1 L_M U),\, \chi_1 U) \\ &= -c_1(L_1\chi_1 U,\, \partial_x(\chi_1 U)) - c_1((L_M - L_1)\chi_1 U,\, \partial_x(\chi_1 U)) - c_1([\chi_1,\, L_M]U,\, \partial_x(\chi_1 U)) \\ &= \frac{1}{2} c_1([\partial_x,\, L_1]\chi_1 U,\, \chi_1 U) + I_{3R} \end{aligned}$$

with the remainder I_{3R} defined by

$$\begin{aligned} I_{3R} &= \frac{1}{2} c_1([\partial_x,\, L_M - L_1]\chi_1 U,\, \chi_1 U) - c_1([\chi_1,\, L_M]U,\, \partial_x(\chi_1 U)) \\ &= -\frac{1}{2} c_1([\partial_x,\, \mathcal{P}_M - \mathcal{P}_1 - a_M + a_1]\chi_1 U_1,\, \chi_1 U_1) + c_1((\partial_x v_M - \partial_x v_1)\chi_1 U_1,\, \partial_x(\chi_1 U_2)) \\ &\quad + \frac{1}{2} c_1([\partial_x,\, G_M - G_1]\chi_1 U_2,\, \chi_1 U_2) - c_1([\chi_1,\, L_M]U,\, \partial_x(\chi_1 U)). \end{aligned}$$

Note that the structure of I_{3R} is very similar to the one of the I_{2R} term above (basically ∂_t is replaced by ∂_x in the commutators) and hence by using the same arguments as above, we get

$$|I_{3R}| \le \frac{1}{h} C(|U(t)|_{X^0}^2 + |U_2(t)|_{L^2}^2).$$

In a symmetric way, we also have for the I_4 term in (5.10) that

$$I_{4R} = \frac{1}{2} c_2([\partial_x,\, L_2]\chi_2 U,\, \chi_2 U) + I_{4R}$$

with

$$|I_{4R}| \le \frac{1}{h} C(|U(t)|_{X^0}^2 + |U_2(t)|_{L^2}^2).$$

Since the solitary waves have the dependence

$$Q_1 = Q_1(x - c_1 t), \quad Q_2 = Q_2(x - h - c_2 t),$$

we have that

$$[\partial_t,\, L_i] = -c_i[\partial_x,\, L_i].$$

This yields the crucial cancellation

$$I_2 + I_3 + I_4 = I_{2R} + I_{3R} + I_{4R}$$

and hence, we obtain the estimate

$$|I_2 + I_3 + I_4| \le \frac{1}{h} C(|U(t)|_{X^0}^2 + |U_2(t)|_{L^2}^2).$$

By using integration by parts and (5.8), we also easily get that

$$|I_5| + |I_6| \le \frac{C}{h}\big(|U_1|_{H^1} + |U_2|_{L^2}\big).$$

Summing up the estimates, we get from (5.10) that

$$\frac{1}{2}\frac{d}{dt}E_1(U(t)) \le \frac{1}{h}C(|U(t)|^2_{X^0} + |U_2(t)|^2_{L^2}). \tag{5.14}$$

The next step will be to get a minoration of $E_1(U(t))$. By using the decomposition of unity (5.7) again, one has

$$\begin{aligned} E_1(U(t)) &= (L_M U,\, U) - c_1(A\chi_1 U,\, \chi_1 U) - c_2(A\chi_2 U,\, \chi_2 U) \\ &= \sum_{i=1,2}(L_M\chi_i U,\, \chi_i U) - c_1(A\chi_1 U,\, \chi_1 U) - c_2(A\chi_2 U,\, \chi_2 U) + \sum_{i=1,2}([L_M,\, \chi_i]\chi_i U,\, U) \\ &:= II_1 + II_2 + II_3 \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} II_1 &= (L_M\chi_1 U,\, \chi_1 U) - c_1(A\chi_1 U,\, \chi_1 U), \\ II_2 &= (L_M\chi_2 U,\, \chi_2 U) - c_2(A\chi_2 U,\, \chi_2 U) \quad \text{and} \quad II_3 = \sum_{i=1,2}([L_M,\, \chi_i]\chi_i U,\, U). \end{aligned}$$

We shall first handle II_1. We note that

$$\begin{aligned} II_1 &= (L_1\chi_1 U,\, \chi_1 U) - c_1(A\chi_1 U,\, \chi_1 U) + ((L_M - L_1)\chi_1 U,\, \chi_1 U) \\ &= (\tilde{L}_1\chi_1 U,\, \chi_1 U) - ((L_M - L_1)\chi_1 U,\, \chi_1 U) \end{aligned} \tag{5.16}$$

where

$$\tilde{L}_1 = \begin{pmatrix} -\mathcal{P}_1 + g + v_1\partial_x Z_1 + \partial_t Z_1 & (v_1 - c_1)\partial_x \\ -\partial_x((v_1 - c_1)\cdot) & G[\eta_1] \end{pmatrix}.$$

Noticing that $Q_1 = Q_1(x - c_1 t)$ and so $\partial_t Z_1 = -c_1\partial_x Z_1$, we can rewrite $\tilde{L}_1$ as

$$\tilde{L}_1 = \begin{pmatrix} -\mathcal{P}_1 + g + (v_1 - c_1)\partial_x Z_1 & (v_1 - c_1)\partial_x \\ -\partial_x((v_1 - c_1)\cdot) & G[\eta_1] \end{pmatrix}.$$

Note that the operators $\tilde{L}_1$ is the same operator as L_{c_1} studied in section 3.2 except that it coefficients depends on $Q_1 = Q_1(x - c_1 t)$, we have

$$\mathcal{T}_{c_1 t}\tilde{L}_1 = L_{c_1}\mathcal{T}_{c_1 t}$$

where $\mathcal{T}_{x_0}$ is the translation operator

$$(\mathcal{T}_{x_0}U)(x) = U(x + x_0).$$

Since $\mathcal{T}_{ct}$ is an isometry on L^2 and X^0, Proposition 3.6 applies to $\tilde{L}_1$. Let us use the notation that $Q_1'(x) = \partial_x Q_{c_1}(x)$ and define

$$\bar{U}^1(t,y) = U^1(t, y + c_1 t) = (\mathcal{T}_{c_1 t}U^1)(t,y) = \chi_1(t, y + c_1 t)U(t, y + c_1 t).$$

Thanks to Proposition 3.11, we can use the decomposition

$$\bar{U}^1(t,y) = \alpha_1(t)JR_1Q_1'(y) + \beta_1(t)R_1Q_1'(y) + W^1(t,y) \tag{5.17}$$

such that W^1 satisfies

$$\big(W^1,\, (\eta_1', 0)^t\big) = 0, \quad (W^1,\, JR_1Q_1') = 0.$$

Note that we use again the short-hand $R_i = R_{c_i}$. Since by definitions, we have

$$\big(\tilde{L}_1U^1, U^1\big) = \big(\mathcal{T}_{-c_1 t}L_{c_1}\mathcal{T}_{c_1 t}U^1, U^1\big) = \big(L_{c_1}\mathcal{T}_{c_1 t}U^1, \mathcal{T}_{c_1 t}U^1\big) = \big(L_{c_1}\bar{U}^1, \bar{U}^1\big),$$

we get from Proposition 3.11, that

$$(\tilde{L}_1U^1,\, U^1) \ge c_0|W^1|^2_{X^0} - C|\alpha_1|^2.$$

Moreover, as in the estimate for I_{2R} above, we also get from Lemma 5.4 that

$$|((L_M - L_1)\chi_1 U,\, \chi_1 U)| \le \frac{1}{h} C(|U|_{X^0}^2 + |U|_{L^2}^2).$$

Consequently, we get from the two previous estimates and (5.16) that

$$II_1 \ge c_0 |W^1|_{X^0}^2 - C|\alpha_1|^2 - \frac{1}{h} C\big(|U|_{X^0}^2 + |U|_{L^2}^2\big).$$

In a symmetric way, we can set

$$\bar{U}^2(t,y) = (\mathcal{T}_{c_2 t + h} U^2)(t,y) = U^2(t, y + h + c_2 t) = \chi_2(t, y + h + c_2 t) U(t, y + h + c_2 t)$$

and use the decomposition

$$\bar{U}^2(t,y) = \alpha_2(t) J R_2 Q_2'(y) + \beta_2(t) R_2 Q_2'(y) + W^2(t,y) \tag{5.18}$$

with

$$(W^2,\, (\eta_2', 0)^t) = 0, \quad (W^2,\, J R_2 Q_2') = 0$$

to obtain that

$$II_2 \ge c_0 |W^2|_{X^0}^2 - C|\alpha_2|^2 - \frac{1}{h} C\big(|U|_{X^0}^2 + |U|_{L^2}^2\big).$$

Moreover, we also have from Lemma 5.4 the estimate

$$|II_3| \le \frac{1}{h} C(|U|_{X^0}^2 + |U_2|_{L^2}^2)$$

by using again that the commutator always involves at least one derivative of χ_i.

In view of the decomposition (5.15), we have thus obtained that

$$E_1(U(t)) \ge c_0(|W^1|_{X^0}^2 + |W^2|_{X^0}^2) - C(|\alpha_1|^2 + |\alpha_2|^2) - \frac{1}{h} C(|U|_{X^0}^2 + |U_2|_{L^2}^2). \tag{5.19}$$

In order to conclude, we still need to estimate $|U_2|_{L^2}$, $|\alpha_i|$ and $|\beta_i| (i = 1, 2)$.

For the L^2 norm, let us choose $\kappa(D)$ where $\kappa \in C_0^\infty(\mathbb{R})$ and $\kappa(\xi) = 1$ around $\xi = 0$. From the linear system (5.5), we have

$$\partial_t U_2 = (P_M - a_M) U_1 - g U_1 - v_M \partial_x U_2,$$

and thus we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\kappa(D) U_2|_{L^2}^2 &= (\kappa(D)\partial_t U_2,\, \kappa(D) U_2) \\ &= (\kappa(D)(P_M - a_M) U_1,\, \kappa(D) U_2) - g(\kappa(D) U_1,\, \kappa(D) U_2) \\ &\quad -(\kappa(D)(v_M \partial_x U_2),\, \kappa(D) U_2). \end{aligned}$$

By using that κ is compactly supported, this yields

$$\frac{1}{2}\frac{d}{dt}|\kappa(D) U_2|_{L^2}^2 \le C\big(|U_1|_{H^1} + |\mathfrak{P} U_2|_{L^2}\big)\big(|\mathfrak{P} U_2|_{L^2} + |\kappa(D) U_2|_{L^2}\big)$$

and hence, we obtain from the Young inequality that

$$\frac{1}{2}\frac{d}{dt}|\kappa(D) U_2|_{L^2}^2 \le C(\epsilon^{-1}|U|_{X^0}^2 + \epsilon |\kappa(D) U_2|_{L^2}^2) \tag{5.20}$$

where ϵ is a small constant to be fixed later.

To estimate α_i, we use the decompositions (5.17), (5.18) of U^i $(i=1,2)$. We have

$$\begin{aligned}
\frac{d}{dt}\alpha_1 &= \frac{1}{|JRQ_1'|_{L^2}^2}\partial_t(\bar\chi_1(t)\bar U(t),\, JR_1Q_1') \\
&= \frac{1}{|JRQ_1'|_{L^2}^2}\Big(((\partial_t\bar\chi_1(t))\bar U(t),\, JR_1Q_1') + (\bar\chi_1(t)(J\overline{L_M U} + c_1\overline{\partial_x U})(t),\, JR_1Q_1')\Big) \\
&= \frac{1}{|JRQ_1'|_{L^2}^2}\Big(((\partial_t\bar\chi_1(t))\bar U(t),\, JR_1Q_1') + (\bar\chi_1(t)J(\overline{L_M U} - \overline{L_1 U})(t),\, JR_1Q_1') \\
&\quad +(\bar\chi_1(t)(J\overline{L_1 U} + c_1\overline{\partial_x U})(t),\, JR_1Q_1')\Big)
\end{aligned}$$

with the notation that $\bar f(t,x) = f(t, x+c_1t)$. We also have

$$\begin{aligned}
&(\bar\chi_1(t)(J\overline{L_1 U} + c_1\overline{\partial_x U})(t),\, JR_1Q_1') \\
&= (\bar\chi_1(t)J(\overline{L_1 U} - c_1J\overline{\partial_x U})(t),\, JR_1Q_1') = (\bar\chi_1(t)J\tilde L_1[Q_1(y)]\bar U(t),\, JR_1Q_1') \\
&= \big([\bar\chi_1(t),\, J\tilde L_1[Q_1(y)]]\bar U(t),\, JR_1Q_1'\big) - (\bar\chi_1(t)\bar U(t),\, \tilde L_1[Q_1(y)]J^2R_1Q_1') \\
&= \big([\bar\chi_1(t),\, J\tilde L_1[Q_1(y)]]\bar U(t),\, JR_1Q_1'\big).
\end{aligned}$$

Indeed, to pass from the second to the third line, we have used the crucial cancellation

$$\big(\overline{L}_1 - c_1J\partial_x\big)J^2R_1Q_1' = -\tilde L_1R_1Q_1' = -L_{c_1}R_{c_1}Q_{c_1}' = 0.$$

By using again Lemma 5.4 to estimate the other terms, we thus obtain

$$\Big|\frac{d}{dt}\alpha_1\Big| \le \frac{1}{h}C(|U|_{X^0} + |U_2|_{L^2}). \tag{5.21}$$

In a symmetric way, we also get for $|\alpha_2|$ that

$$\Big|\frac{d}{dt}\alpha_2\Big| \le \frac{1}{h}C(|U|_{X^0} + |U_2|_{L^2}). \tag{5.22}$$

We still need the estimates of $|\beta_1|$ and $|\beta_2|$. By using (3.40), we note that we can write

$$\beta_1 = \tilde\beta_1 - \alpha_1\big(JR_1Q_1', (\eta_1', 0)^t\big)$$

where

$$\tilde\beta_1 = \frac{\big(\bar U^1, (\eta_1', 0)^t\big)}{|\eta'|_{L^2}^2}.$$

In particular, thanks to (5.21), we obtain that

$$\Big|\frac{d}{dt}\beta_1\Big| \le \Big|\frac{d}{dt}\tilde\beta_1\Big| + \frac{1}{h}C(|U|_{X^0} + |U_2|_{L^2}). \tag{5.23}$$

To estimate $\tilde\beta_1$, we directly compute

$$\begin{aligned}
\frac{d}{dt}\tilde\beta_1(t) &= \frac{1}{|\eta_1'|_{L^2}^2}\partial_t(\bar U^1(t),\, (\eta_1', 0)^t) \\
&= \frac{1}{|\eta_1'|_{L^2}^2}\Big(((\partial_t\bar\chi_1(t))\bar U(t),\, (\eta_1', 0)^t) + (\bar\chi_1(t)(\overline{\partial_t U} + c_1\overline{\partial_x U})(t),\, (\eta_1', 0)^t)\Big) \\
&= \frac{1}{|\eta_1'|_{L^2}^2}\Big(((\partial_t\bar\chi_1(t))\bar U(t),\, (\eta_1', 0)^t) + (\bar\chi_1(t)J\overline{L_M U}(t),\, (\eta_1', 0)^t) \\
&\quad +c_1(\bar\chi_1(t)\overline{\partial_x U}(t),\, (\eta_1', 0)^t)\Big)
\end{aligned}$$

where in particular

$$(\bar{\chi}_1(t)J\overline{L_M U}(t),\, (\eta_1',0)^t) \;=\; (\bar{\chi}_1(t)J(\overline{L_M U}-\overline{L_1 U})(t),\, (\eta_1',0)^t) + ([\bar{\chi}_1(t),\, J\bar{L}_1]\bar{U},\, (\eta_1',0)^t) \\ +(J\overline{L_1 U^1}(t),\, (\eta_1',0)^t)\big)$$

and

$$c_1(\bar{\chi}_1(t)\overline{\partial_x U}(t),\, (\eta_1',0)^t) = -c_1((\partial_x\bar{\chi}_1(t)\bar{U}(t),\, (\eta_1',0)^t) + c_1(\overline{\partial_x U^1}(t),\, (\eta_1',0)^t).$$

Therefore, we get

$$\big|\frac{d}{dt}\tilde{\beta}_1(t)\big| \;\le\; C\big|\big(J\tilde{L}_1\overline{U}^1,(\eta_1',0)\big)\big| + \frac{1}{h}C(|U|_{X^0}+|U_2|_{L^2}).$$

To conclude, we use the decomposition (5.17), since $\tilde{L}_1 R_1 Q_1' = 0$, we have

$$\big|J\tilde{L}_1\overline{U}^1,(\eta_1',0)\big)\big| \le C\big(|\alpha_1| + |W^1|_{X^0}\big)$$

and hence, we get from (5.23) that

$$\big|\frac{d}{dt}\beta_1(t)\big| \le C\Big(\frac{1}{h}(|U|_{X^0}+|U_2|_{L^2}) + |\alpha_1| + |W^1|_{X^0}\Big). \tag{5.24}$$

Similarly we have

$$\big|\frac{d}{dt}\beta_2(t)\big| \le C\left(\frac{1}{h}(|U|_{X^0}+|U_2|_{L^2}) + |\alpha_2| + |W^2|_{X^0}\right). \tag{5.25}$$

From (5.21), (5.22), (5.24), (5.25), we finally obtain

$$\frac{d}{dt}|\alpha_i|^2 \le C|\alpha_i|\frac{1}{h}(|U|_{X^0}+|U_2|_{L^2}), \quad i=1,\,2 \tag{5.26}$$

$$\frac{d}{dt}|\beta_i|^2 \le C|\beta_i|\Big(\frac{1}{h}(|U|_{X^0}+|U_2|_{L^2}) + |\alpha_i| + |W^i|_{X^0}\Big), \quad i=1,\,2. \tag{5.27}$$

To combine (5.26), (5.27), (5.14), (5.20) and (5.19) in an efficient way, we define a weighted energy

$$\tilde{E}_1(U(t)) = \frac{1}{2}h^{\frac{1}{2}}E_1(U(t)) + \frac{1}{2}(|\beta_1(t)|^2 + |\beta_2(t)|^2) + Ch^{\frac{1}{2}}(|\alpha_1(t)|^2 + |\alpha_2(t)|^2) \tag{5.28}$$

From (5.14) and (5.20), (5.26), (5.27), we obtain

$$\begin{aligned}
&\frac{d}{dt}\tilde{E}_1(U(t)) \\
&\le h^{-\frac{1}{4}}C\big[h^{-\frac{1}{4}}(|U|_{X^0}^2 + |\kappa(D)U_2|_{L^2}^2) + h^{-\frac{3}{4}}(|\beta_1|+|\beta_2|)(|U|_{X^0}+|U_2|_{L^2}) \\
&\qquad + h^{\frac{1}{4}}(|\beta_1|+|\beta_2|)(|\alpha_1|+|\alpha_2|+|W^1|_{X^0}+|W^2|_{X^0}) + h^{-\frac{1}{4}}(|\alpha_1|+|\alpha_2|)(|U|_{X^0}+|U_2|_{L^2})\big] \\
&\le h^{-\frac{1}{4}}C\big[h^{\frac{1}{2}}(|W^1|_{X^0}^2 + |W^2|_{X^0}^2) + (|\beta_1|^2+|\beta_2|^2) + h^{\frac{1}{2}}(|\alpha_1|^2+|\alpha_2|^2) + h^{-\frac{1}{4}}|U_2|_{L^2}^2)\big].
\end{aligned}$$

To get the last line, we have used the decompositions of U^1, U^2, (5.17), (5.18) and Lemma 5.6. Now, let us define

$$F(t) = h^{\frac{1}{2}}(|W^1|_{X^0}^2 + |W^2|_{X^0}^2) + (|\beta_1|^2+|\beta_2|^2) + h^{\frac{1}{2}}(|\alpha_1|^2+|\alpha_2|^2)$$

and integrate the estimate for $\tilde{E}_1(U(t))$ with respect to time from 0 to t, we obtain

$$\tilde{E}_1(U(t)) \le \tilde{E}_1(U(0)) + h^{-\frac{1}{4}}C\int_0^t (F(s) + h^{-\frac{1}{4}}|\kappa(D)U_2(s)|_{L^2}^2)ds,$$

by using that $|U_2|_{L^2} \leq C(|\kappa(D)U_2|_{L^2} + |U|_{X^0})$. On the other hand, from the definition of $\tilde{E}_1(U)$ in (5.28) and the estimate (5.19), we also have

$$\begin{aligned}
\tilde{E}_1(U(t)) &\geq \frac{1}{2}h^{\frac{1}{2}}c_0(|W^1|^2_{X^0} + |W^2|^2_{X^0}) + \frac{1}{2}(|\beta_1|^2 + |\beta_2|^2) \\
&\quad + h^{\frac{1}{2}}c_0(|\alpha_1|^2 + |\alpha_2|^2) - h^{-\frac{1}{2}}C(\sum_{i=1,2}|U^i|^2_{X^0} + |\kappa(D)U_2|^2_{L^2}) \\
&\geq \bar{c}_0 F(t) - h^{-\frac{1}{2}}C|\kappa(D)U_2|^2_{L^2}
\end{aligned}$$

for some $\overline{c}_0 > 0$. This yields

$$F(t) - h^{-\frac{1}{2}}C|\kappa(D)U_2(t)|^2_{L^2} \leq h^{\frac{1}{2}}C|U(0)|^2_{X^0} + h^{-\frac{1}{4}}C\int_0^t (F(s) + h^{-\frac{1}{4}}|\kappa(D)U_2(s)|^2_{L^2})ds. \qquad (5.29)$$

By taking the integral of (5.20) with respect to time, we also have

$$|\kappa(D)U_2(t)|^2_{L^2}(t) \leq |U_2(0)|^2_{L^2} + C\int_0^t (\epsilon^{-1}F(s) + \epsilon|\kappa(D)U_2(s)|^2_{L^2})ds \qquad (5.30)$$

where the constant ϵ will be fixed later. We take the sum of (5.29) and (5.30) multiplied with some $\lambda > 0$ that will be also chosen later to get that

$$\begin{aligned}
&F(t) + (\lambda - h^{-\frac{1}{2}}C)|\kappa(D)U_2(t)|^2_{L^2} \\
&\leq h^{\frac{1}{2}}C|U(0)|^2_{X^0} + \lambda|U_2(0)|^2_{L^2} + Ch^{-\frac{1}{4}}\int_0^t (F(s) + h^{-\frac{1}{4}}|\kappa(D)U_2(s)|^2_{L^2})ds \\
&\quad + \lambda C\int_0^t (\epsilon^{-1}F(s) + \epsilon|\kappa(D)U_2(s)|^2_{L^2})ds.
\end{aligned}$$

Let us choose λ and ϵ in order to satisfy the following conditions

$$\lambda - h^{-\frac{1}{2}}C \geq \lambda/2, \quad Ch^{-\frac{1}{4}} \leq \gamma/4, \quad h^{-\frac{1}{4}} \leq \lambda/2, \quad \lambda C\epsilon^{-1} \leq \gamma/4 \quad \text{and} \quad \epsilon^2 \leq \lambda/2,$$

where $\gamma = \epsilon_0(c_2 - c_1)$ comes from Proposition 4.1. One can choose for example $\lambda = \frac{\gamma^2}{64C^2}$, $\epsilon = \frac{\sqrt{\lambda}}{2}$ and h large enough to get our final energy estimate

$$F(t) + \frac{\lambda}{2}|\kappa(D)U_2(t)|^2_{L^2} \leq h^{\frac{1}{2}}C|U(0)|^2_{X^0} + \lambda|U_2(0)|^2_{L^2} + \frac{\gamma}{4}\int_0^t (F(s) + \frac{\lambda}{2}|\kappa(D)U_2(s)|^2_{L^2})ds.$$

Applying Gronwall's inequality, we get

$$F(t) + \frac{\lambda}{2}|\kappa(D)U_2(t)|^2_{L^2} \leq (h^{\frac{1}{2}}C|U(0)|^2_{X^0} + \lambda|U_2(0)|^2_{L^2})e^{\gamma t/4}, \quad \text{for any} \quad t \geq 0.$$

From the definition of $F(t)$ and Lemma 5.6, we get that there exist constants c_0 and $\bar{c}_0$ such that

$$\begin{aligned}
F(t) &= h^{\frac{1}{2}}(|W^1|^2_{X^0} + |W^2|^2_{X^0}) + (|\beta_1|^2 + |\beta_2|^2) + h^{\frac{1}{2}}(|\alpha_1|^2 + |\alpha_2|^2) \\
&\geq c_0\sum_{i=1,2}|U^i(t)|^2_{X^0} \geq \big(1 - \frac{C}{h}\big)\bar{c}_0|U(t)|^2_{X^0} - \frac{c_0}{h}|\kappa(D)U_2(t)|^2_{L^2}.
\end{aligned}$$

By choosing h such that $\frac{c_0}{h} < \frac{\lambda}{4}$, we find

$$F(t) \geq \frac{1}{2}\bar{c}_0|U|^2_{X^0} - \frac{\lambda}{4}|\kappa(D)U_2|^2_{L^2}.$$

Summing up the estimates above, we finally obtain

$$\frac{\bar{c}_0}{2}|U|^2_{X^0} + \frac{\lambda}{4}|\kappa(D)U_2|^2_{L^2} \leq (h^{\frac{1}{2}}C|U(0)|^2_{X^0} + \lambda|U_2(0)|^2_{L^2})e^{\epsilon_0(c_2-c_1)t/4}, \quad \text{for any} \quad t \geq 0,$$

that is to say

$$|U|_{X^0}^2 + |U_2|_{L^2}^2 \leq C(h^{\frac{1}{2}}|U(0)|_{X^0}^2 + |U_2(0)|_{L^2}^2)e^{\epsilon_0(c_2-c_1)t/4}, \qquad t \geq 0.$$

This ends the proof of Proposition 5.5.

5.3. **Proof of Theorem 5.2.** We shall prove Theorem 5.2 by induction. The $k = 0$ case was already obtained in Proposition 5.5. For the sake of clarity, before the general induction argument, we shall explain the proof for the $k = 1$ energy estimate.

Order-1 energy estimates. Note that since the coefficients in the linear system (5.5) depend on t and x, neither ∂_t nor ∂_x has a nice commutation property with the equation. We shall thus use the operator

$$D(\partial) = \partial_t + c_1\chi_1^2\partial_x + c_2\chi_2^2\partial_x \tag{5.31}$$

in order to take derivatives of the equation. This will take into account the fact that the solitary waves depend on $x - c_i t$.

By applying this operator on both sides of system (5.5), we obtain for $D(\partial)U$ the system:

$$\partial_t D(\partial)U - JL_M D(\partial)U = [\partial_t,\, D(\partial)]U - J[L_M,\, D(\partial)]U. \tag{5.32}$$

A further computation shows that

$$[\partial_t,\, D(\partial)]U = c_1(\partial_t\chi_1^2)\partial_x U + c_2(\partial_t\chi_2^2)\partial_x U,$$

and

$$\begin{aligned} J[L_M,\, D(\partial)]U =& J[L_M,\, \partial_t]U + J\sum_{i=1,2} c_i[L_M,\, \chi_i^2\partial_x]U \\ =& J\sum_{i=1,2}\big(\chi_i^2[L_M - L_i,\, \partial_t]U + \chi_i^2[L_i,\, \partial_t]U + c_i[L_M,\, \chi_i^2]\partial_x U + c_i\chi_i^2[L_M - L_i,\, \partial_x]U \\ &+ \chi_i^2[L_i,\, c_i\partial_x]U\big) \\ =& J\sum_{i=1,2}\big(\chi_i^2[L_M - L_i,\, \partial_t]U + c_i[L_M,\, \chi_i^2]\partial_x U + c_i\chi_i^2[L_M - L_i,\, \partial_x]U\big) := JSU. \end{aligned} \tag{5.33}$$

The crucial fact that we have used in the above computation is the cancellation:

$$J\sum_{i=1,2}\chi_i^2[L_i,\, \partial_t]U + J\sum_{i=1,2}\chi_i^2[L_i,\, c_i\partial_x]U = J\sum_{i=1,2}\chi_i^2[L_i,\, \partial_t + c_i\partial_x]U = 0$$

We can thus rewrite the system (5.32) for $D(\partial)U$ as

$$\partial_t D(\partial)U = JL_M D(\partial)U - JSU + [\partial_t,\, D(\partial)]U := JL_M D(\partial)U + F_1(U) \tag{5.34}$$

where we will take $F_1(U) = -JSU + [\partial_t,\, D(\partial)]U$ as the source term.

From Proposition 5.5, we have for the fundamental solution $S_M(t,\tau)$ of the linear equation (5.5) the estimate

$$\|S_M(t,\tau)\|_{X^0\cap L^2 \to X^0\cap L^2} \leq h^{\frac{1}{4}}Ce^{\epsilon_0(c_2-c_1)(t-\tau)/4}, \qquad \forall t \geq \tau \geq 0.$$

Consequently, by using Duhamel's Formula, we can rewrite (5.32) as

$$D(\partial)U(t) = S_M(t,0)(D(\partial)U)(0) + \int_0^t S_M(t,\tau)F_1(\tau)d\tau,$$

so we get

$$|D(\partial)U(t)|_{X^0} + |D(\partial)U_2|_{L^2}$$
$$\leq h^{\frac{1}{4}} C\big[e^{\epsilon_0(c_2-c_1)t/4}(|U(0)|_{X^1} + \sum_{\alpha=0,1} |\partial_t^\alpha U_2(0)|_{L^2}) + \int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|F_1(\tau)|_{X^0} + |F_1(\tau)|_{L^2})d\tau\big]. \tag{5.35}$$

We still need to estimate the source term F_1. We shall first use the elliptic regularity of the leading spatial operators in (5.5) to prove that time derivatives control higher order space derivatives.

For any norm $\|\cdot\|$ on x dependent vectors, we use the notation

$$\|\langle\partial_t\rangle^k U\| = \sum_{0\leq l\leq k} \|\partial_t^l U\|.$$

Lemma 5.8. *Any smooth solution of (5.5) satisfies the following a priori estimates:*

$$\forall l \geq 0,\ m\geq 0,\ \exists C_{k,l}, \quad \big|\partial_t^l(U_1,U_2)\big|_{H^{m+\frac{5}{2}}\times H^{m+2}} \leq C_{l,m} |\langle\partial_t\rangle^{l+1}(U_1,U_2)|_{H^{m+1}\times H^{m+\frac{1}{2}}}. \tag{5.36}$$

Proof. This is an elliptic regularity result. Let us start with the case $l=0$. We first rewrite (5.5) under the form:

$$\left\{\begin{array}{l} G_M U_2 = \partial_t U_1 + \partial_x(v_M U_1), \\ \mathcal{P}_M U_1 = \partial_t U_2 + (a_M+g)U_1 + v_M\partial_x U_2. \end{array}\right. \tag{5.37}$$

The operator $\mathcal{P}_M$ is an elliptic operator of order two, therefore, by classical elliptic regularity results, we immediately obtain from the second line of the above system

$$|U_1|_{H^{m+\frac{5}{2}}} \leq C_m\Big(|\partial_t U_2|_{H^{m+\frac{1}{2}}} + |\partial_x U_2|_{H^{m+\frac{1}{2}}} + |U_1|_{H^m}\Big)$$

and thus by the interpolation inequality

$$|U_1|_{H^m} \leq \delta |U_1|_{H^{m+\frac{5}{2}}} + C_\delta |U_1|_{L^2},$$

we actually obtain

$$|U_1|_{H^{m+\frac{5}{2}}} \leq C_m\Big(|\partial_t U_2|_{H^{m+\frac{1}{2}}} + |U_2|_{m+\frac{3}{2}} + |U_1|_{L^2}\Big). \tag{5.38}$$

In a similar way, since the surface η_M is smooth, the Dirichlet-Neumann operator $G[\eta_M]$ is an elliptic operator of order one. Actually, we have

$$G[\eta_M] = |D_x| + R(t,x,D_x) \tag{5.39}$$

where R is a pseudo differential operator of order zero. We refer to [17], [28] for the proof. Note that we are in dimension one thus the principal symbol is very simple. Consequently, we also get from the first equation of (5.37)

$$|U_2|_{H^{m+2}} \leq C_m\Big(|\partial_t U_1|_{H^{m+1}} + |U_1|_{H^{m+2}} + |U_2|_{L^2}\Big). \tag{5.40}$$

By using again that

$$|U_1|_{H^{m+2}} \leq \delta|U_1|_{H^{m+\frac{5}{2}}} + C_\delta|U_1|_{L^2}, \quad |U_2|_{m+\frac{3}{2}} \leq \delta |U_2|_{H^{m+2}} + C_\delta |U_2|_{L^2},$$

we get from (5.38), (5.40) by choosing δ sufficiently small that

$$|U_1|_{H^{m+\frac{5}{2}}} + |U_2|_{H^{m+2}} \leq C_m\Big(|\partial_t U_1|_{H^{m+1}} + |\partial_t U_2|_{H^{m+\frac{1}{2}}} + |U|_{L^2}\Big).$$

We have thus proven (5.36) for $l=0$.

We then proceed by induction on l. Assume that the result is proven for $l-1$ time derivatives. Let us apply ∂_t^l to (5.37). From the second line, we get

$$\mathcal{P}_M\partial_t^l U_1 = \partial_t^{l+1}U_2 + \partial_t^l\big((a_M+g)U_1\big) + \partial_t^l\big(v_M\partial_x U_2\big) - [\partial_t^l, \mathcal{P}_M]U_1 := F^1$$

and we observe that the right hand-side satisfies the estimate

$$|F^1|_{H^{m+\frac12}} \le C_{l,m}\Big(|\partial_t^{l+1}U_2|_{H^{m+\frac12}} + |\langle\partial_t\rangle^l U_2|_{H^{m+\frac32}} + |\langle\partial_t\rangle^{l-1}U_1|_{H^{m+\frac52}}\Big)$$

and thus we get by elliptic regularity that

$$|\partial_t^l U_1|_{H^{m+\frac52}} \le C_{l,m}\Big(|\partial_t^{l+1}U_2|_{H^{m+\frac12}} + |\langle\partial_t\rangle^l U_2|_{H^{m+\frac32}} + |\langle\partial_t\rangle^{l-1}U_1|_{H^{m+\frac52}}\Big). \tag{5.41}$$

In a similar way, for the first line of (5.37), we get

$$G_M\partial_t^l U_2 = \partial_t^{l+1}U_1 + \partial_t^l\partial_x(v_M U_1) - [\partial_t^l, G_M]U_2 := F^2.$$

By using Proposition 3.1 (6) to compute the commutator $[\partial_t^l, G_M]U_2$, we obtain for the right hand side the estimate

$$|F^2|_{H^{m+1}} \le C_{l,m}\Big(|\partial_t^{l+1}U_1|_{H^{m+1}} + |\langle\partial_t\rangle^l U_1|_{H^{m+2}} + |\langle\partial_t\rangle^{l-1}U_2|_{H^{m+2}}\Big).$$

Consequently, from the ellipticity of the Dirichlet-Neumann operator, we also get

$$|\partial_t^l U_2|_{H^{m+2}} \le C_{l,m}\Big(|\partial_t^{l+1}U_1|_{H^{m+1}} + |\langle\partial_t\rangle^l U_1|_{H^{m+2}} + |\langle\partial_t\rangle^{l-1}U_2|_{H^{m+2}}\Big). \tag{5.42}$$

By combining (5.41) and (5.42) we get

$$\begin{aligned}|\partial_t^l(U_1,U_2)|_{H^{m+\frac52}\times H^{m+2}} \le C_{l,m}\Big(&|\partial_t^{l+1}(U_1,U_2)|_{H^{m+1}\times H^{m+\frac12}} \\ &+ |\partial_t^l(U_1,U_2)|_{H^{m+2}\times H^{m+\frac32}} + |\langle\partial_t\rangle^{l-1}(U_1,U_2)|_{H^{m+\frac52}\times H^{m+2}}\Big).\end{aligned}$$

To conclude, it suffices to use the interpolation inequality

$$|\partial_t^l(U_1,U_2)|_{H^{m+2}\times H^{m+\frac32}} \le \delta|\partial_t^l(U_1,U_2)|_{H^{m+\frac52}\times H^{m+2}} + C_\delta|\partial_t^l U|_{L^2}$$

and the induction assumption. This ends the proof of Lemma 5.8. □

As a consequence of Lemma 5.8, we get the following inequalities that we will use many times:

- By using the lemma with $l=0$, $m=0$, we have

  $$|\partial_x U|_{H^{\frac32}\times H^1} \le C\big(|\langle\partial_t\rangle U|_{H^1\times H^{\frac12}}\big)$$

  and hence, since, we can use again that

  $$|\partial_x U|_{H^1\times H^{\frac12}} \le \epsilon|\partial_x U|_{H^{\frac32}\times H^1} + C_\epsilon|U|_{L^2} \tag{5.43}$$

  we get that

  $$|\partial_x U|_{X^0} \le \epsilon(|\partial_t U|_{X^0} + |\partial_t U_2|_{L^2}) + C_\epsilon(|U|_{X^0} + |U|_{L^2}) \tag{5.44}$$

  for any $\epsilon > 0$.
- We can also use Lemma 5.8, with $l=\alpha$, $m=\beta-1$ for any α and β such that $\alpha+\beta=k$ and $\beta\ge1$. We obtain

  $$|\langle\partial_t\rangle^\alpha U|_{H^{\beta+\frac32}\times H^{\beta+1}} \le C_k|\langle\partial_t\rangle^{\alpha+1}U|_{H^\beta\times H^{\beta-\frac12}}$$

  and we can iterate the process to obtain

  $$|\partial_t^\alpha\partial_x^\beta U|_{H^{\frac32}\times H^1} \le |\langle\partial_t\rangle^\alpha U|_{H^{\beta+\frac32}\times H^{\beta+1}} \le C_k|\langle\partial_t\rangle^k U|_{H^1\times H^{\frac12}}.$$

  Thanks to (5.43), we thus obtain

  $$|\partial_t^\alpha\partial_x^\beta U|_{X^0} \le \epsilon\big(|\partial_k U|_{X^0} + |\partial_t^k U_2|_{L^2}\big) + C_\epsilon\big(|U|_{X^{k-1}} + |\langle\partial_t\rangle^{k-1}U_2|_{L^2}\big) \tag{5.45}$$

  for $\alpha+\beta=k$, $\beta\ge1$.

We can come back to the estimate for the source term $F_1(U(t))$ in the right hand side of (5.35). By using the expression of $F_1(U(t))$ in (5.33), we get

$$\begin{aligned}|F_1(U)|_{X^0} &\leq |JSU|_{X^0} + |[\partial_t,\, D(\partial)]U|_{X^0}\\ &\leq |J\sum_{i=1,2}\chi_i^2[L_M - L_i, \partial_t]U|_{X^0} + |J\sum_{i=1,2} c_i[L_M,\, \chi_i^2]\partial_x U|_{X^0}\\ &\quad + |J\sum_{i=1,2} c_i\chi_i^2[L_M - L_i,\, \partial_x]U|_{X^0} + |[\partial_t,\, D(\partial)]U|_{X^0}.\end{aligned}$$

By using again Lemma 5.7 and Lemma 5.4 as before, we obtain

$$|F_1(U)|_{X^0} \leq \frac{1}{h}C(|(\partial_x U_2,\, \partial_x^2 U_1)^t|_{X^0} + |\partial_x U|_{X^0} + |U|_{X^0}),$$

and hence, by using Lemma 5.8, we find

$$|F_1(U)|_{X^0} \leq \frac{1}{h}C(|\partial_t U|_{X^0} + |U|_{X^0} + |\partial_t U_2|_{L^2} + |U_2|_{L^2}).$$

In a similar way, we also have

$$|F_1(U)|_{L^2} \leq \frac{1}{h}C(|\partial_t U|_{X^0} + |U|_{X^0} + |\partial_t U_2|_{L^2} + |U_2|_{L^2}).$$

Going back to (5.35), we obtain

$$\begin{aligned}&|D(\partial)U(t)|_{X^0} + |D(\partial)U_2(t)|_{L^2}\\ &\leq h^{\frac{1}{4}}Ce^{\epsilon_0(c_2-c_1)t/4}(|U(0)|_{X^1} + |U_2(0)|_{\tilde{H}^1})\\ &\quad + h^{-\frac{3}{4}}C\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|\partial_t U(\tau)|_{X^0} + |U(\tau)|_{X^0} + |\partial_t U_2(\tau)|_{L^2} + |U_2(\tau)|_{L^2})d\tau.\end{aligned}$$

It remains to deduce an estimate of the time and space derivatives from the previous estimate that gives only a control of the $D(\partial)$ derivatives of U. We first observe that

$$\begin{aligned}&|D(\partial)U(t)|_{X^0} + |D(\partial)U_2(t)|_{L^2}\\ &\geq |\partial_t U(t)|_{X^0} + |\partial_t U_2(t)|_{L^2} - \sum_{i=1,2} c_i(|\chi_i^2\partial_x U(t)|_{X^0} + |\chi_i^2\partial_x U_2(t)|_{L^2})\\ &\geq |\partial_t U(t)|_{X^0} + |\partial_t U_2(t)|_{L^2} - C(|\partial_x U|_{X^0} + |\partial_x U|_{L^2}).\end{aligned}$$

We know from (5.44) that

$$|D(\partial)U(t)|_{X^0} + |D(\partial)U_2(t)|_{L^2} \geq |\partial_t U(t)|_{X^0} + |\partial_t U_2(t)|_{L^2} - C(|U(t)|_{X^0} + |U_2(t)|_{L^2}).$$

We have thus proven that

$$\begin{aligned}&|\partial_t U(t)|_{X^0} + |\partial_t U_2(t)|_{L^2} - C(|U(t)|_{X^0} + |U_2(t)|_{L^2})\\ &\leq h^{\frac{1}{4}}Ce^{\epsilon_0(c_2-c_1)t/4}(|U(0)|_{X^1} + |U_2(0)|_{L^2} + |\partial_t U_2(0)|_{L^2})\\ &\quad + h^{-\frac{3}{4}}C\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|\partial_t U(\tau)|_{X^0} + |U(\tau)|_{X^0} + |U_2(\tau)|_{L^2} + |\partial_t U_2(\tau)|_{L^2})d\tau.\end{aligned} \tag{5.46}$$

From Proposition 5.5, we have

$$|U(\tau)|_{X^0} + |U_2(\tau)|_{L^2} \leq h^{\frac{1}{4}}C(|U(0)|_{X^0} + |U_2(0)|_{L^2})e^{\epsilon_0(c_2-c_1)\tau/4}, \qquad \text{for} \quad \tau \in [0,\, t],$$

and therefore

$$h^{-\frac{3}{4}}C\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|U(\tau)|_{X^0} + |U_2(\tau)|_{L^2})d\tau \leq t\,h^{-\frac{1}{2}}C(|U(0)|_{X^0} + |U_2(0)|_{L^2})e^{\epsilon_0(c_2-c_1)t/4},$$

consequently, we finally deduce from (5.46) that

$$\begin{aligned}|\partial_t U(t)|_{X^0}+|\partial_t U_2(t)|_{L^2} &\leq h^{\frac{1}{4}}C(1+h^{-\frac{3}{4}}t)e^{\epsilon_0(c_2-c_1)t/4}(|U(0)|_{X^1}+|U_2(0)|_{L^2}+|\partial_t U_2(0)|_{L^2})\\ &\quad +h^{-\frac{3}{4}}C\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|\partial_t U(\tau)|_{X^0}+|\partial_t U_2(\tau)|_{L^2})d\tau.\end{aligned}$$

Note that we could avoid the additional algebraic growth but that we do not need to refine.

The Gronwall's inequality yields that there exists a constant C_1 such that

$$|\partial_t U(t)|_{X^0}+|\partial_t U_2(t)|_{L^2}\leq h^{\frac{1}{4}}C_1(1+h^{-\frac{3}{4}}t)e^{\epsilon_0(c_2-c_1)t/4+h^{-\frac{3}{4}}C_1 t}(|U(0)|_{X^1}+\sum_{\alpha=0,1}|\partial_t^\alpha U_2(0)|_{L^2}).$$

This yields the desired estimate for the time derivative by taking h large enough. We finally deduce from (5.44) that the same estimate also holds for $|\partial_x U(t)|_{X^0}$. This ends the proof of the order-1 energy estimate.

Higher-order energy estimates. We will do this by an induction argument. First of all, assume that we already have the estimates for $|U|_{X^{k-1}}$ and $|\kappa(D)\partial_t^\alpha U_2|_{L^2}$ with $k\geq 2$ and $\alpha\leq k-1$:

$$\begin{aligned}&|U|_{X^{k-1}}+\sum_{\alpha\leq k-1}|\partial_t^\alpha U_2|_{L^2}\\ &\leq h^{\frac{1}{4}}C_{k-1}(|U(0)|^2_{X^{k-1}}+\sum_{\alpha\leq k-1}|\partial_t^\alpha U_2(0)|^2_{L^2})(1+h^{-\frac{3}{4}}t)^{k-1}e^{\epsilon_0(c_2-c_1)t/4+h^{-\frac{3}{4}}C_1 t}.\end{aligned}$$

As previously, we get start with the estimate of $|D^k(\partial)U|_{X^0}$. One can write the system solved by $D^k(\partial)U$ as

$$\partial_t D^k(\partial)U = JL_M D(\partial)^k U + F_k(U) \tag{5.47}$$

where

$$F_k(U)=\sum_{i=0}^{k-1}D(\partial)^i\big(J[D(\partial),\, L_M]+[\partial_t,\, D(\partial)]\big)D(\partial)^{k-1-i}U$$

will be considered as the source term.

From the Duhamel formula, we find

$$D(\partial)^k U(t) = S_M(t,0)D(\partial)^k U(0)+\int_0^t S_M(t,\tau)F_k(U(\tau))d\tau$$

and therefore, we again obtain from Proposition 5.5 that

$$\begin{aligned}&|D(\partial)^k U(t)|_{X^0\cap L^2}\\ &\leq h^{\frac{1}{4}}Ce^{\epsilon_0(c_2-c_1)t/4}(|U(0)|_{X^k}+\sum_{\alpha\leq k-1}|\partial_t^\alpha U_2(0)|_{L^2})+h^{\frac{1}{4}}C\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}|F_k(\tau)|_{X^0\cap L^2}d\tau.\end{aligned} \tag{5.48}$$

It remains to estimate the source term $F_k(t)$. We have

$$
\begin{aligned}
&|F_k(U)|_{X^0}\\
&\leq \sum_{i=0}^{k-1}\big(|JD(\partial)^i[D(\partial),\, L_M]D(\partial)^{k-1-i}U|_{X^0}+|D(\partial)^i[\partial_t,\, D(\partial)]D(\partial)^{k-1-i}U|_{X^0}\big)\\
&\leq \sum_{i=0}^{k-1}\sum_{j=1,2}\big[|JD(\partial)^i\chi_j^2[L_M-L_j,\partial_t]D(\partial)^{k-1-i}U|_{X^0}+c_j|JD(\partial)^i[L_M,\,\chi_j^2]\partial_x D(\partial)^{k-1-i}U|_{X^0}\\
&\quad +c_j|JD(\partial)^i\chi_j^2[L_M-L_j,\,\partial_x]D(\partial)^{k-1-i}U|_{X^0}\big]+\sum_{i=0}^{k-1}|D(\partial)^i[\partial_t,\, D(\partial)]D(\partial)^{k-1-i}U|_{X^0}.
\end{aligned}
$$

Consequently, by using again the same arguments to estimate the commutators and the observation (already used in the order-1 estimate) that the commutator $[D(\partial),\, L_M]=S$ (computed in (5.33)), acts like $\frac{1}{h}L_M$, we can get that

$$
|F_k(U)|_{X^0}+|F_k(U)|_{L^2}\leq \frac{1}{h}C_k\sum_{\substack{\alpha+\beta=k\\ \alpha\leq k-1}}(|(\partial_t^\alpha\partial_x^\beta U_2,\, \partial_t^\alpha\partial_x^{\beta+1}U_1)^t|_{X^0}+|\partial_t^\alpha\partial_x^\beta U|_{X^0})+\frac{1}{h}C|U|_{X^{k-1}},
$$

and so by using Lemma 5.8 again we arrive at

$$
|F_k(U)|_{X^0}+|F_k(U)|_{L^2}\leq \frac{1}{h}C_k(|\partial_t^k U|_{X^0}+|U|_{X^{k-1}}+\sum_{\alpha\leq k}|\partial_t^\alpha U_2|_{L^2}).
$$

Going back to (5.48) one has

$$
\begin{aligned}
&|D(\partial)^kU(t)|_{X^0}+|D(\partial)^kU_2(t)|_{L^2}\\
&\leq h^{\frac14}Ce^{\epsilon_0(c_2-c_1)t/4}(|U(0)|_{X^k}+\sum_{\alpha\leq k}|\partial_t^\alpha U_2(0)|_{L^2}))\\
&\quad +h^{-\frac34}C_k\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|\partial_t^kU(\tau)|_{X^0}+|U(\tau)|_{X^{k-1}}+\sum_{\alpha\leq k}|\partial_t^\alpha U_2(\tau)|_{L^2})d\tau.
\end{aligned}
$$

For the left-hand side, we observe that

$$
|D(\partial)^kU(t)|_{X^0}+|D(\partial)^kU_2(t)|_{L^2}\geq|\partial_t^kU(t)|_{X^0}+|\partial_t^kU_2(t)|_{L^2}-C(\sum_{\substack{\alpha+\beta\leq k\\ \alpha\leq k-1}}|\partial_t^\alpha\partial_x^\beta U(t)|_{X^0}+|U(t)|_{X^{k-1}}),
$$

which together with (5.45) allows to get

$$
|D(\partial)^kU(t)|_{X^0}+|D(\partial)^kU_2(t)|_{L^2}\geq|\partial_t^kU(t)|_{X^0}+|\partial_t^kU_2|_{L^2}-C(|U(t)|_{X^{k-1}}+\sum_{\alpha\leq k-1}|\partial_t^\alpha U_2|_{L^2}).
$$

Consequently, by using the induction assumption to estimate $|U|_{X^{k-1}}+\sum_{\alpha\leq k-1}|\partial_t^\alpha U_2|_{L^2}$, we find

$$
\begin{aligned}
|\partial_t^kU(t)|_{X^0}+|\partial_t^kU_2|_{L^2}\quad\leq\quad & h^{\frac14}C_{k-1}(1+h^{-\frac34}t)^{k-1}e^{\epsilon_0(c_2-c_1)t/4+h^{-\frac34}C_1t}(|U(0)|_{X^k}+\sum_{\alpha\leq k}|\partial_t^\alpha U_2(0)|_{L^2})\\
&+h^{-\frac34}C_k\int_0^t e^{\epsilon_0(c_2-c_1)(t-\tau)/4}(|\partial_t^kU(\tau)|_{X^0}+|\partial_t^kU_2(\tau)|_{L^2})d\tau.
\end{aligned}
$$

Using the Gronwall's inequality as before yields

$$
|\partial_t^kU(t)|_{X^0}+|\partial_t^kU_2|_{L^2}\leq h^{\frac14}C_k(|U(0)|_{X^k}+\sum_{\alpha\leq k}\partial_t^\alpha|U_2(0)|)(1+h^{-\frac34}t)^ke^{\epsilon_0(c_2-c_1)t/4+h^{-\frac34}C_1t}.
$$

Finally, by using again Lemma 5.8 and by taking h large enough such that $h^{-\frac{3}{4}}C_1 \leq \epsilon_0(c_2-c_1)/4$ we conclude that

$$|U(t)|_{X^k} + \sum_{\alpha\leq k} |\partial_t^\alpha U_2|_{L^2} \leq h^{\frac{1}{4}} C_k(|U(0)|_{X^k} + \sum_{\alpha\leq k} |\partial_t^\alpha U_2(0)|)(1+\epsilon_0(c_2-c_1)t)^k e^{\epsilon_0(c_2-c_1)t/2}, \quad \forall t \geq 0.$$

This completes the proof of Theorem 5.2.

Remark 5.9. *Note that in the proof of Theorem 5.2, the only information about the positions of the solitary waves are in the interaction estimates given in Lemma 5.4. These localization estimates are true on any interval of time for which the distance between the centers of the solitary waves is bigger than h which is considered as a large parameter. We can use this remark to get an estimate for $S_M(t,\tau)$ for $0 \leq t \leq \tau$. Indeed by using again the reversibility symmetry of the water waves system and the symmetries of the solitary waves of Theorem 1.1, consider $U(t,x)$ the solution of* (5.5) *with initial data at $t=\tau$, then*

$$\tilde{U}(t,x) = \big(U_1(\tau-t,-x), -U_2(\tau-t,-x)\big)^t$$

is still a solution on $[0,\tau]$ of (5.5) *with $M(t,x) = Q_1(x-c_1t) + Q_2(x-c_2t-h)$ replaced by*

$$\tilde{M}(t,x) = Q_1(x+c_1\tau-c_1t) + Q_2(x+c_2\tau+h-c_2t)$$

and with initial data for $\tilde{U}$ at $t=0$. Consequently, we see that on $[0,\tau]$, the center of the solitary waves are located at $x=x_1=-c_1|t-\tau|$ and $x=x_2=-c_2|t-\tau|-h$. Consequently, the slow solitary wave Q_1 is now located on the right. Nevertheless, we observe that for $t\in[0,\tau]$, we still have

$$x_1 - x_2 \geq (c_2-c_1)|t-\tau| + h \geq h$$

and thus the solitary waves are still at least at distance h uniformly in τ. Consequently, we still get as in Theorem 5.2 that

$$|\tilde{U}(t)|_{E^k} \leq h^{\frac{1}{4}} C_k |\tilde{U}(0)|_{H^{s(k)}} \big(1+\epsilon_0(c_2-c_1)t^k\big) e^{\frac{\epsilon_0 t}{2}}, \quad \forall t \in [0,\tau].$$

This yields in the original time variable

$$|U(t)|_{E^k} \leq h^{\frac{1}{4}} C_k |U(\tau)|_{H^{s(k)}} \big(1+\epsilon_0(c_2-c_1)(\tau-t)^k\big) e^{\frac{\epsilon_0(\tau-t)}{2}}, \quad \forall t \in [0,\tau].$$

By combining with Theorem 5.2, we thus obtain that the fundamental solution $S_M(t,\tau)$ of system (5.5) *enjoys the estimate*

$$|S_M(t,\tau)U|_{E^k} \leq h^{\frac{1}{4}} C_k |U|_{H^{s(k)}} \big(1+\epsilon_0(c_2-c_1)|t-\tau|^k)\big) e^{\frac{\epsilon_0|t-\tau|}{2}}, \quad \forall t,\tau \geq 0. \tag{5.49}$$

5.4. **Proof of Proposition 5.1: Construction of the approximate solution.** Now we can go back to the study of the linear systems solved by V_l $(1 \leq l \leq N)$ (5.2), (5.3) in order to prove Proposition 5.1.

We first note from the fact that (5.4) and (5.5) are equivalent via the transformation $U(t) = RV(t)$, with R invertible, we get from Theorem 5.2 and (5.6)

$$|S_M^\Lambda(t,\tau)V|_{E^k} \leq h^{\frac{1}{4}} C_k(\epsilon_0)|V|_{H^{s(k)}}(1+|t-\tau|^k)e^{\epsilon_0(c_2-c_1)|t-\tau|/2}, \quad \forall t,\tau \geq 0 \tag{5.50}$$

where $S_M^\Lambda(t,\tau)$ is the fundamental solution of the system (5.4).

Let us go back to the construction of the approximate solution $V(t,x) = \sum_{l=1}^N \delta^l V_l(t,x)$ with $\delta = e^{-\epsilon_0 h}$. For $V_1(t)$, we have to solve

$$\partial_t V_1 - J\Lambda[M]V_1 = -\tilde{R}_M,$$

where the right hand side satisfies (see 5.1) the estimate

$$|\tilde{R}_M|_{E^k} \leq C_k e^{-\epsilon_0(c_2-c_1)t}, \quad \forall t \geq 0, k \in \mathbb{N}.$$

We choose the solution

$$V_1(t,x) = -\int_t^\infty S_M^\Lambda(t,\tau)\tilde{R}_M(\tau)d\tau.$$

From the estimate (5.50) of the fundamental solution, V_1 is well-defined and satisfies the estimate

$$\begin{aligned}
|V_1(t)|_{E^k} &\leq h^{\frac{1}{4}} C_k \int_t^\infty (1+|t-\tau|)^k e^{\epsilon_0(c_2-c_1)(\tau-t)/2} \sum_{l=0}^k \|\partial_t^l \tilde{R}_M(\tau)\|_{H^{s(k-l)}} d\tau \\
&\leq h^{\frac{1}{4}} C_k \int_t^\infty (1+|t-\tau|)^k e^{\epsilon_0(c_2-c_1)(\tau-t)/2} e^{-\epsilon_0(c_2-c_1)\tau} d\tau \\
&\leq h^{\frac{1}{4}} C_{k,1}(\epsilon_0) e^{-\epsilon_0(c_2-c_1)t}, \quad \text{for} \quad t \geq 0.
\end{aligned}$$

For the general case of V_l $(2 \leq l \leq N)$, we have to solve

$$\partial_t V_l - J\Lambda[M]V_l = \sum_{p=2}^{l} \sum_{\substack{1\leq l_1,\dots,l_p\leq N \\ l_1+\dots+l_p=l}} \frac{1}{p!} D^p\mathcal{F}[M](V_{l_1},\dots,V_{l_p}) := R_l(t,x). \tag{5.51}$$

We shall use an induction argument. Let us assume that we already have already built V_j, $1 \leq j \leq l-1$ that satisfy the estimate

$$|V_j(t)|_{E^k} \leq h^{\frac{2j-1}{4}} C_{k,j}(\epsilon_0) e^{-j\epsilon_0(c_2-c_1)t}, \quad \forall t \geq 0.$$

We get for the right-hand side of (5.51) that

$$\begin{aligned}
|R_l(t,x)|_{E^k} &= \Big|\sum_{p=2}^{l} \sum_{\substack{1\leq l_1,\dots,l_p\leq N \\ l_1+\dots+l_p=l}} \frac{1}{p!} D^p\mathcal{F}[M](V_{l_1},\dots,V_{l_p})\Big|_{E^k} \\
&\leq h^{\frac{2l-1}{4}} C_{k,l}(\epsilon_0) e^{-l\epsilon_0(c_2-c_1)t}.
\end{aligned}$$

Taking

$$V_l(t,x) = -\int_t^\infty S_M^\Lambda(t,\tau) R_l(\tau) d\tau$$

as a solution, we get thanks to (5.50)

$$\begin{aligned}
|V_l(t)|_{E^k} &\leq h^{\frac{2l-1}{4}} C_{k,l}(\epsilon_0) \int_t^\infty (1+\epsilon_0(c_2-c_1)(\tau-t))^k e^{\epsilon_0(c_2-c_1)(\tau-t)/2} e^{-l\epsilon_0(c_2-c_1)\tau} d\tau \\
&\leq h^{\frac{2l-1}{4}} C_{k,l}(\epsilon_0) e^{-l\epsilon_0(c_2-c_1)t}, \quad \forall t \geq 0.
\end{aligned}$$

This ends the proof of Proposition 5.1. □

6. The nonlinear problem

After the study of the approximate solution U^a of water-wave system, we need to consider the remainder solution $U^R = U - U^a$ where U is the solution of the water-wave system (1.6) and U^R satisfies

$$\begin{cases} \partial_t U^R = \mathcal{F}(U^a + U^R) - \mathcal{F}(U^a) - R_{ap}, \quad t > 0, \\ U^R(0) \text{ to be fixed later} \end{cases} \tag{6.1}$$

Proposition 6.1. *Let $m \geq 2$, $U^a \in W^{m+s}_{[0,\infty)}$ and $R_{ap} \in X^{m+3}_{[0,\infty)}$. There exists a solution $U^R = (\eta^R, \varphi^R)^t \in L^\infty([0,\infty), H^{m+4} \times H^{m+\frac{7}{2}})$ for (6.1) with a fixed initial value $U^R(0)$ such that*

$$H - \|\eta^a\|_{L^\infty} - \|\eta^R\|_{L^\infty} > 0$$

and
$$|U^R(t)|_{H^{m+4}\times H^{m+\frac{7}{2}}} \leq C_{N,m} h^{\frac{2N+1}{4}} \delta^{N+1} e^{-(N+1)\epsilon_0(c_2-c_1)t}, \quad \textit{for any} \quad t\in[0,\infty).$$

Proof. The proof is left to the reader, it suffices to use the same arguments as in the proof of Theorem 1.3. □

End of the proof of Theorem 1.2. Since we have already shown the global existence of the remainder solution U^R, we know that there exits a (semi-) global solution $U(t,x) = U^a(t,x) + U^R(t,x) = M(t,x) + V(t,x) + U^R(t,x)$ to the water-wave system (1.5). It only remains to describe the asymptotic behavior of $U(t)$ when t tends to $+\infty$. Since

$$U(t) = M(t,x) + \sum_{l=1}^{N} \delta^l V_l(t) + U^R(t)$$

with $\delta = e^{-\epsilon_0 h}$ and with V_l and U^R that satisfy the estimates from Proposition 5.1 and Proposition 6.1, we have

$$|V_l(t)|_{E^s} \leq h^{\frac{2l-1}{4}} C_{N,s} e^{-l\epsilon_0(c_2-c_1)t}, \quad |U^R(t)|_{X^{s+3}} \leq h^{\frac{2N+1}{4}} C_{N,s} \delta^{N+1} e^{-(N+1)\epsilon_0(c_2-c_1)t}.$$

This gives in particular

$$|V_l(t)|_{H^s} \leq h^{\frac{2l-1}{4}} C_{N,s} e^{-l\epsilon_0(c_2-c_1)t}, \quad |U^R(t)|_{H^s} \leq h^{\frac{2N+1}{4}} C_{N,s} \delta^{N+1} e^{-(N+1)\epsilon_0(c_2-c_1)t}$$

for any $t \in [0,\infty)$. We thus get that

$$\lim_{t\to+\infty} |U(t) - M(t)|_{H^s} = 0.$$

This ends the proof of Theorem 1.2. □

7. Appendix: Proof of Lemma 5.7

7.1. **Proof of** (5.12)**.** We shall only prove the case $i = 1$, the case $i = 2$ can then be obtained by symmetric arguments. Let us recall that as in (3.2), (3.1), the Dirichlet-Neumann operator $G[\eta]$ can be defined by:

$$G[\eta]u = \partial_n^P u^b|_{z=0}$$

where u^b satisfies the elliptic system on the flat strip $\mathcal{S} = \mathbb{R}\times[-1,0]$

$$\begin{cases} \nabla_{x,z}\cdot P\nabla_{x,z}u^b = 0, & \text{in} \quad \mathcal{S} \\ u^b|_{z=0} = u, & \partial_n^P u^b|_{z=-1} = 0. \end{cases} \tag{7.1}$$

with the notations

$$P = P[\eta] = \begin{pmatrix} H+\eta & -(z+1)\partial_x\eta \\ -(z+1)\partial_x\eta & \frac{1+(z+1)^2(\partial_x\eta)^2}{H+\eta} \end{pmatrix}$$

and $\partial_n^P = \mathbf{n}\cdot P\nabla_{x,z}$ where $\mathbf{n} = -e_z$ is the outward unit normal to the boundary $z=-1$. Note that $\partial_n^P|_{z=-1} = -\frac{1}{H+\eta}\partial_z$.

One can see by the Green's Formula that

$$(G[\eta]u, v) = \int_{\mathcal{S}} P\nabla_{x,z}u^b\cdot\nabla_{x,z}v^b$$

and so we have

$$([\partial_t, G[\eta]]u,\, u) = \int_{\mathcal{S}} \partial_t P\nabla_{x,z}u^b\cdot\nabla_{x,z}u^b - 2\int_{\mathcal{S}} P\nabla_{x,z}((\partial_t u)^b - \partial_t u^b)\cdot\nabla_{x,z}u^b.$$

In the following we shall use the notations $(j = M, 1, 2)$

$$G[\eta_j] = G_j, \quad P[\eta_j] = P_j \quad \text{and} \quad u^b[\eta_j] = u_j^b.$$

We also set $u = \chi_1 U_2$ for the sake of convenience. Then we can write

$$\begin{aligned}
&([\partial_t,\, G_M - G_1]u,\, u)\\
&= \int_{\mathcal{S}} \partial_t (P_M - P_1)\nabla_{x,z} u_M^b \cdot \nabla_{x,z} u_M^b + \int_{\mathcal{S}} \partial_t P_1 \nabla_{x,z}(u_M^b - u_1^b)\cdot \nabla_{x,z} u_M^b\\
&\quad + \int_{\mathcal{S}} \partial_t P_1 \nabla_{x,z} u_1^b \cdot \nabla_{x,z}(u_M^b - u_1^b) - 2\int_{\mathcal{S}} (P_M - P_1)\nabla_{x,z} u_M^b \cdot \nabla_{x,z}((\partial_t u)_M^b - \partial_t u_M^b)\\
&\quad - 2\int_{\mathcal{S}} P_1 \nabla_{x,z}[(\partial_t u)_M^b - \partial_t u_M^b - (\partial_t u)_1^b + \partial_t u_1^b]\cdot \nabla_{x,z} u_M^b\\
&\quad - 2\int_{\mathcal{S}} P_1 \nabla_{x,z}((\partial_t u)_1^b - \partial_t u_1^b)\cdot \nabla_{x,z}(u_M^b - u_1^b).
\end{aligned} \tag{7.2}$$

Let us recall that the solitary wave η_2 satisfies the exponential decay estimate

$$|\partial_x^\alpha \eta_2(x - h - c_2 t)| \le C_\alpha e^{-d(1+|x-h-c_2 t|^2)^{\frac{1}{2}}} \quad \text{for} \quad x, t \in \mathbb{R}$$

Consequently, we shall use the weight f defined by

$$f(t,x) = e^{-\epsilon(1+|x-h-c_2 t|^2)^{\frac{1}{2}}}$$

where $\epsilon \in [0, d]$ will be chosen sufficiently small.

From (7.2), we first get the estimate

$$\begin{aligned}
\big|([\partial_t,\, G_M - G_1]u,\, u)\big| &\le C\Big(\|f\nabla_{x,z} u_M^b\|_{L^2(\mathcal{S})} + \|\nabla_{x,z}(u_M^b - u_1^b)\|_{L^2(\mathcal{S})}\\
&\quad + \|\nabla_{x,z}\big((\partial_t u)_M^b - \partial_t u_M^b - (\partial_t u)_1^b + \partial_t u_1^b\big)\|_{L^2(\mathcal{S})}\Big)\cdot \Big(\|\nabla_{x,z} u_M^b\|_{L^2(\mathcal{S})} + \|\nabla_{x,z} u_1^b\|_{L^2(\mathcal{S})}\\
&\quad + \|\nabla_{x,z}\big((\partial_t u)_M^b - \partial_t u_M^b\big)\|_{L^2(\mathcal{S})} + \|\nabla_{x,z}\big((\partial_t u)_1^b - \partial_t u_1^b\big)\|_{L^2(\mathcal{S})}\Big).
\end{aligned} \tag{7.3}$$

From the elliptic problem (7.1), we first get the estimates (this follows for example by using the decomposition (3.6) of the solution, we refer to [4], [24] for example)

$$\|\nabla_{x,z} u_i^b\|_{L^2(\mathcal{S})} \le C|\mathfrak{P}u|_{L^2(\mathbb{R}^2)} \le C(|\mathfrak{P}U_2|_{L^2}^2 + |U_2|_{L^2}^2), \quad i = M,\, 1, \tag{7.4}$$

where in the last step we used (5.11). To conclude, we still need the estimates for $f u_M^b$, $u_M^b - u_1^b$, $(\partial_t u)_j^b - \partial_t u_j^b$ and $(\partial_t u)_M^b - \partial_t u_M^b - (\partial_t u)_1^b + \partial_t u_1^b$ with $j = M, 1$. We shall deal with them one by one.

1) Estimate for $e^{-\epsilon(1+|x-h-c_2 t|^2)^{\frac{1}{2}}} u_M^b$. As in the proof of Proposition 3.2, we get that $e^{-\epsilon(1+|x-h-c_2 t|^2)^{\frac{1}{2}}} u_M^b$ solves the elliptic equation

$$\begin{cases} \nabla_{x,z}\cdot P_M \nabla_{x,z}(f u_M^b) = -[f,\, \nabla_{x,z}\cdot P_M \nabla_{x,z}]u_M^b, \quad \text{in} \quad \mathcal{S},\\ f u_M^b|_{z=0} = f u, \quad \partial_n^{P_M}(f u_M^b)|_{z=-1} = -[f,\, \partial_n^{P_M}]u_M^b|_{z=-1} = 0 \end{cases}$$

We introduce the decomposition

$$f u_M^b = m(z, |D|)(f u) + v$$

where $m(z, |D|)$ is the Fourier multiplier $\frac{\cosh\,(|D|(z+1))}{\cosh\,|D|}$. We get for v the system

$$\begin{cases} \nabla_{x,z}\cdot P_M \nabla_{x,z} v = -\nabla_{x,z}\cdot P_M \nabla_{x,z}(m(f u)) - [f,\, \nabla_{x,z}\cdot P_M \nabla_{x,z}]u_M^b, \quad \text{in} \quad \mathcal{S},\\ v|_{z=0} = 0, \quad \partial_z v|_{z=-1} = 0. \end{cases}$$

From an energy estimate (as in the proof of Proposition 3.2), we obtain that

$$\|\nabla_{x,z}v\|^2_{L^2(\mathcal{S})} \leq C\Big(\|\nabla_{x,z}(m(z,|D|)(fu))\|_{L^2(\mathcal{S})}\|\nabla_{x,z}v\|_{L^2(\mathcal{S})} + (\epsilon^2\|fu^b_M\|_{L^2(\mathcal{S})} + \epsilon\|\nabla_{x,z}(fu^b_M)\|_{L^2(\mathcal{S})})(\|v\|_{L^2(\mathcal{S})} + \|\nabla_{x,z}v\|_{L^2(\mathcal{S})})\Big) \quad (7.5)$$

where in particular we have use the fact that

$$|\nabla_{x,z}f| \lesssim \epsilon f.$$

Next, we can use the Poincaré inequality in the strip $\mathcal{S}$ which yields

$$\|v\|_{L^2(\mathcal{S})} \leq C\|\nabla_{x,z}v\|_{L^2(\mathcal{S})}$$

and the fact that $u = \chi_1 U_2$ which yields thanks to (5.11) and Lemma 5.4

$$\|m(z,|D|)(fu)\|_{L^2(\mathcal{S})} + \|\nabla_{x,z}(m(z,|D|)(fu))\|_{L^2(\mathcal{S})} \leq \frac{1}{h}C(|\mathfrak{P}U_2|_{L^2} + |U_2|_{L^2})$$

to obtain from (7.5) by taking ϵ small enough that

$$\|v\|^2_{L^2} + \|\nabla_{x,z}v\|^2_{L^2} \leq \frac{1}{h}C(|\mathfrak{P}U_2|^2_{L^2} + |U_2|^2_{L^2}).$$

This yields

$$\|\nabla_{x,z}(fu^b_M)\|^2_{L^2} + \|fu^b_M\|^2_{L^2} \leq \frac{1}{h}C(|\mathfrak{P}U_2|^2_{L^2} + |U_2|^2_{L^2}). \quad (7.6)$$

2) Estimate for $u^b_M - u^b_1$. Let us set

$$\mathbf{g} = -(P_M - P_1)\nabla_{x,z}u^b_M,$$

we get that $u^b_M - u^b_1$ solves

$$\begin{cases} \nabla_{x,z}\cdot P_1\nabla_{x,z}(u^b_M - u^b_1) = \nabla_{x,z}\cdot\mathbf{g}, \quad \text{in} \quad \mathcal{S}, \\ u^b_M - u^b_1|_{z=0} = 0, \quad \partial_z(u^b_M - u^b_1)|_{z=-1} = 0 \end{cases}$$

The standard energy estimate for this problem yields

$$\|\nabla_{x,z}(u^b_M - u^b_1)\|^2_{L^2} \leq C\|\mathbf{g}\|^2_2 \leq C(\|fu^b_M\|^2_{L^2} + \|\nabla_{x,z}(fu^b_M)\|^2_{L^2}),$$

which results (combined with the Poincaré inequality and (7.6)) in

$$\|u^b_M - u^b_1\|^2_{L^2} + \|\nabla_{x,z}(u^b_M - u^b_1)\|^2_{L^2} \leq \frac{1}{h}C(|\mathfrak{P}U_2|^2_{L^2} + |U_2|^2_{L^2}).$$

3) Estimate for $(\partial_t u)^b_j - \partial_t u^b_j$. As previously, we can set $\mathbf{g}_j = \partial_t P_j\nabla_{x,z}u^b_j$ with $j = M, 1$, to get that $(\partial_t u)^b_j - \partial_t u^b_j$ satisfies the system

$$\begin{cases} \nabla_{x,z}\cdot P_j\nabla_{x,z}((\partial_t u)^b_j - \partial_t u^b_j) = \nabla_{x,z}\cdot\mathbf{g}_j, \quad \text{in} \quad \mathcal{S}, \\ (\partial_t u)^b_j - \partial_t u^b_j|_{z=0} = 0, \quad \partial_n^{P_M}((\partial_t u)^b_j - \partial_t u^b_j)|_{z=-1} = -\mathbf{e_z}\cdot\mathbf{g}_j|_{z=-1} = 0 \end{cases}$$

and we obtain that

$$\|(\partial_t u)^b_j - \partial_t u^b_j\|^2_{L^2} + \|\nabla_{x,z}((\partial_t u)^b_j - \partial_t u^b_j)\|^2_{L^2} \leq C\|\mathbf{g}_j\|^2_{L^2} \leq C(|\mathfrak{P}U_2|^2_{L^2} + |U_2|^2_{L^2}) \quad (7.7)$$

thanks to (7.4).

4) Estimate for $(\partial_t u)^b_M - \partial_t u^b_M - (\partial_t u)^b_1 + \partial_t u^b_1$. We write $v = (\partial_t u)^b_M - \partial_t u^b_M - (\partial_t u)^b_1 + \partial_t u^b_1$ here and we know from 3) that

$$\begin{cases} \nabla_{x,z}\cdot P_1\nabla_{x,z}v = \nabla_{x,z}\cdot[\mathbf{g}_M - \mathbf{g}_1 - (P_M - P_1)\nabla_{x,z}((\partial_t u)^b_M - \partial_t u^b_M)], \quad \text{in} \quad \mathcal{S}, \\ v|_{z=0} = 0, \quad \partial_n^{P_M}v|_{z=-1} = -\mathbf{e_z}\cdot[\mathbf{g}_M - \mathbf{g}_1 - (P_M - P_1)\nabla_{x,z}((\partial_t u)^b_M - \partial_t u^b_M)]|_{z=-1} \end{cases}$$

From a basic energy estimate, we obtain

$$
\begin{array}{rcl}
\|\nabla_{x,z}v\|_2 & \leq & C(\|(P_M-P_1)\nabla_{x,z}((\partial_t u)^b_M-\partial_t u^b_M)\|_{L^2}+\|\partial_t(P_M-P_1)\nabla_{x,z}u^b_M\|_{L^2} \\
& & +\|\partial_t P_1\nabla_{x,z}(u^b_M-u^b_1)\|_{L^2} \\
& \leq & C[\|f((\partial_t u)^b_M-\partial_t u^b_M)\|_{L^2}+\|\nabla_{x,z}(f(\partial_t u)^b_M-\partial_t u^b_M)\|_{L^2}+\|fu^b_M\|_{L^2} \\
& & +\|\nabla_{x,z}(fu^b_M)\|_{L^2}+\|\nabla_{x,z}(u^b_M-u^b_1)\|_{L^2}]
\end{array}
\tag{7.8}
$$

where we use again $f=e^{-\epsilon(1+|x-h-c_2t|^2)^{\frac{1}{2}}}$. To conclude, we need to estimate $w=f((\partial_t u)^b_M-\partial_t u^b_M)$. We observe that w solves

$$
\left\{\begin{array}{l}
\nabla_{x,z}\cdot P_M\nabla_{x,z}w=-[f,\,\nabla_{x,z}\cdot P_M\nabla_{x,z}]((\partial_t u)^b_M-\partial_t u^b_M)-f\nabla_{x,z}\cdot\partial_t P_M\nabla_{x,z}u^b_M,\quad \text{in}\quad \mathcal{S}, \\
w|_{z=0}=0,\quad \partial_z w|_{z=-1}=0.
\end{array}\right.
$$

We can proceed as in step 1) to obtain that

$$
\|w\|^2_{L^2}+\|\nabla_{x,z}w\|^2_{L^2}\leq C\|f\partial_t P_M\nabla_{x,z}u^b_M\|^2_{L^2}+\frac{1}{h}C(|\mathfrak{P}U_2|^2_{L^2}+|U_2|^2_{L^2})\leq\frac{1}{h}C(|\mathfrak{P}U_2|^2_{L^2}+|U_2|^2_{L^2})
$$

thanks to (7.6). We can thus also obtain by combining the last estimate, (7.8) and the estimates of step 1) and step 2) that

$$
\|v\|^2_{L^2}+\|\nabla_{x,z}v\|^2_{L^2}\leq\frac{1}{h}C(|\mathfrak{P}U_2|^2_{L^2}+|U_2|^2_{L^2}).
$$

Gathering all the previous estimates, we finally obtain (5.12).

7.2. **Proof of** (5.13)**.** Since

$$
\begin{array}{rcl}
([[\partial_t,\,G_M],\,\chi_1]u,\,v) & = & ([\partial_t,\,G_M]\chi_1 u,\,v)-([\partial_t,\,G_M]u,\,\chi_1 v) \\
& = & \partial_t(G_M\chi_1 u,\,v)-(G_M\chi_1 u,\,\partial_t v)-(G_M\partial_t(\chi_1 u),\,v)-\partial_t(\chi_1 G_M u,\,v) \\
& & +((\partial_t\chi_1)G_M u,\,v)+(\chi_1 G_M u,\,\partial_t v)+(\chi_1 G_M\partial_t u,\,v),
\end{array}
$$

we get by using the Green's Formula on the flat strip $\mathcal{S}$ that

$$
\begin{array}{rcl}
([[\partial_t,\,G_M],\,\chi_1]u,\,v) & = & \partial_t\int_{\mathcal{S}}P_M\nabla_{x,z}(\chi_1 u)^b\cdot\nabla_{x,z}v^\dagger-\partial_t\int_{\mathcal{S}}\chi_1 P_M\nabla_{x,z}u^b\cdot\nabla_{x,z}v^\dagger \\
& & -\partial_t\int_{\mathcal{S}}(\nabla_{x,z}\chi_1)\cdot P_M\nabla_{x,z}u^b\cdot v^\dagger-\int_{\mathcal{S}}P_M\nabla_{x,z}(\chi_1 u)^b\cdot\nabla_{x,z}(\partial_t v)^\dagger \\
& & +\int_{\mathcal{S}}\chi_1 P_M\nabla_{x,z}u^b\cdot\nabla_{x,z}(\partial_t v)^\dagger+\int_{\mathcal{S}}(\nabla_{x,z}\chi_1)\cdot P_M\nabla_{x,z}u^b\,(\partial_t v)^\dagger \\
& & -\int_{\mathcal{S}}P_M\nabla_{x,z}(\partial_t(\chi_1 u))^b\cdot\nabla_{x,z}v^\dagger+\int_{\mathcal{S}}(\partial_t\chi_1)P_M\nabla_{x,z}u^b\cdot\nabla_{x,z}v^\dagger \\
& & +\int_{\mathcal{S}}\nabla_{x,z}(\partial_t\chi_1)\cdot P_M\nabla_{x,z}u^b\,v^\dagger+\int_{\mathcal{S}}\chi_1 P_M\nabla_{x,z}(\partial_t u)^b\cdot\nabla_{x,z}v^\dagger \\
& & +\int_{\mathcal{S}}(\nabla_{x,z}\chi_1)\cdot P_M\nabla_{x,z}(\partial_t u)^b\,v^\dagger
\end{array}
$$

where u^b is u^b_M are satisfying (7.1) with η_M, and $v^\dagger = \chi(z|D|)v$ with χ a smooth compactly supported cut-off function such that $\chi(0) = 1$. One can rewrite the above as

$$
\begin{aligned}
&([[\partial_t, G_M], \chi_1]u, v)\\
&= \int_{\mathcal{S}} [\partial_t P_M \nabla_{x,z}(\chi_1 u)^b \cdot \nabla_{x,z} v^\dagger - \partial_t P_M \chi_1 \nabla_{x,z} u^b \cdot \nabla_{x,z} v^\dagger - (\nabla_{x,z}\chi_1) \cdot \partial_t P_M \nabla_{x,z} u^b\, v^\dagger]\\
&\quad + \int_{\mathcal{S}} [P_M \nabla_{x,z}(\partial_t(\chi_1 u)^b - (\partial_t(\chi_1 u))^b) \cdot \nabla_{x,z} v^\dagger - \chi_1 P_M \nabla_{x,z}((\partial_t u)^b - \partial_t u^b) \cdot \nabla_{x,z} v^\dagger\\
&\quad - (\nabla_{x,z}\chi_1) \cdot P_M \nabla_{x,z}((\partial_t u)^b - \partial_t u^b)\, v^\dagger]\\
&:= A_1 + A_2
\end{aligned}
\tag{7.9}
$$

while noticing that $\partial_t v^\dagger = (\partial_t v)^\dagger$. We will deal with A_1 first. We have

$$
\begin{aligned}
A_1 \;=\; & \int_{\mathcal{S}} \partial_t P_M \nabla_{x,z}((\chi_1 u)^b - \chi_1 u^b) \cdot \nabla_{x,z} v^\dagger + \int_{\mathcal{S}} \partial_t P_M (\nabla_{x,z}\chi_1) u^b \cdot \nabla_{x,z} v^\dagger\\
& - \int_{\mathcal{S}} (\nabla_{x,z}\chi_1) \cdot \partial_t P_M \nabla_{x,z} u^b\, v^\dagger,
\end{aligned}
$$

Since $(\chi_1 u)^b - \chi_1 u^b$ solves the equation

$$
\begin{cases}
\nabla_{x,z} \cdot P_M \nabla_{X,z}((\chi_1 u)^b - \chi_1 u^b) = [\chi_1, \nabla_{x,z} \cdot P_M \nabla_{x,z}] u^b, \quad \text{in} \quad \mathcal{S},\\
(\chi_1 u)^b - \chi_1 u^b|_{z=0} = 0 \quad \partial_n^{P_M}((\chi_1 u)^b - \chi_1 u^b)|_{z=-1} = [\chi_1, \partial_n^{P_M}] u^b_{z=-1} = 0,
\end{cases}
$$

by using the definition of χ_1 and (5.8), we again obtain from an energy estimate that

$$
\|\nabla_{x,z}((\chi_1 u)^b - \chi_1 u^b)\|_{L^2} \leq \frac{1}{h} C(|\mathfrak{P}u|_{L^2} + |u|_{L^2}). \tag{7.10}
$$

By using again (5.8), we thus obtain

$$
A_1 \leq \frac{1}{h} C(|\mathfrak{P}u|^2_{L^2} + |\mathfrak{P}v|^2_{L^2} + |u|^2_{L^2} + |v|^2_{L^2}),
$$

where we used the fact that $\|\nabla_{x,z} v^\dagger\|_2 \leq C|\mathfrak{P}v|_{L^2}$. On the other hand, one can rewrite

$$
\begin{aligned}
A_2 \;=\; & \int_{\mathcal{S}} P_M \nabla_{x,z} \Big(\partial_t(\chi_1 u)^b - (\partial_t(\chi_1 u))^b - \chi_1 \partial_t u^b + \chi_1 (\partial_t u)^b\Big) \cdot \nabla_{x,z} v^\dagger\\
& + \int_{\mathcal{S}} P_M (\nabla_{x,z}\chi_1) \cdot \Big[(\partial_t u^b - (\partial_t u)^b) \nabla_{x,z} v^\dagger - \nabla_{x,z}(\partial_t u^b - (\partial_t u)^b)\, v^\dagger\Big].
\end{aligned}
$$

We already have the estimate for $\partial_t u^b - (\partial_t u)^b$ from step 3) of the last subsection, it only remains to estimate $w = \partial_t(\chi_1 u)^b - (\partial_t(\chi_1 u))^b - \chi_1 \partial_t u^b + \chi_1(\partial_t u)^b$. We get for w the equation in $\mathcal{S}$

$$
\begin{aligned}
\nabla_{x,z} \cdot P_M \nabla_{x,z} w = & - \nabla_{x,z} \cdot \partial_t P_M \nabla_{x,z}(\chi_1 u)^b + [\chi_1, \nabla_{x,z} \cdot P_M \nabla_{x,z}](\partial_t u^b - (\partial_t u)^b)\\
& + \chi_1 \nabla_{x,z} \cdot \partial_t P_M \nabla_{x,z} u^b
\end{aligned}
$$

that we can rewrite as

$$
\begin{aligned}
\nabla_{x,z} \cdot P_M \nabla_{x,z} w = & - \nabla_{x,z} \cdot \partial_t P_M \nabla_{x,z}\big((\chi_1 u)^b - \chi_1 u^b\big) + [\chi_1, \nabla_{x,z} \cdot P_M \nabla_{x,z}](\partial_t u^b - (\partial_t u)^b)\\
& + [\chi_1, \nabla_{x,z} \cdot \partial_t P_M \nabla_{x,z}] u^b
\end{aligned}
$$

with the boundary conditions

$$
w|_{z=0} = 0, \quad \partial_n^{P_M} w|_{z=-1} = 0.
$$

We thus get the estimate

$$
\|w\|_{L^2} + \|\nabla_{x,z} w\|_{L^2} \leq C\Big(\|\nabla_{x,z}\big((\chi_1 u)^b - \chi_1 u^b\big)\|_{L^2} + \frac{1}{h}\|\partial_t u^b - (\partial_t u)^b\|_{H^1} + \frac{1}{h}\|u^b\|_{H^1}\Big).
$$

Consequently, by using (7.10), (7.7) and (7.4), we obtain

$$\|w\|_{L^2} + \|\nabla_{x,z} w\|_{L^2} \leq \frac{1}{h}(|\mathfrak{P}u|_{L^2} + |u|_{L^2})$$

and hence

$$|A_2| \leq \frac{1}{h}(|\mathfrak{P}u|_{L^2} + |u|_{L^2})(|\mathfrak{P}v|_{L^2} + |v|_{L^2}).$$

Gathering the estimates for A_1 and A_2, we end the proof by substituting $u = \chi_1 U_2$ and $v = U_2$ into the estimates. □

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